BRIDGES TO CLASSROOM MATHEMATICS

A joint project of COMAP, TERC, and the University of Chicago
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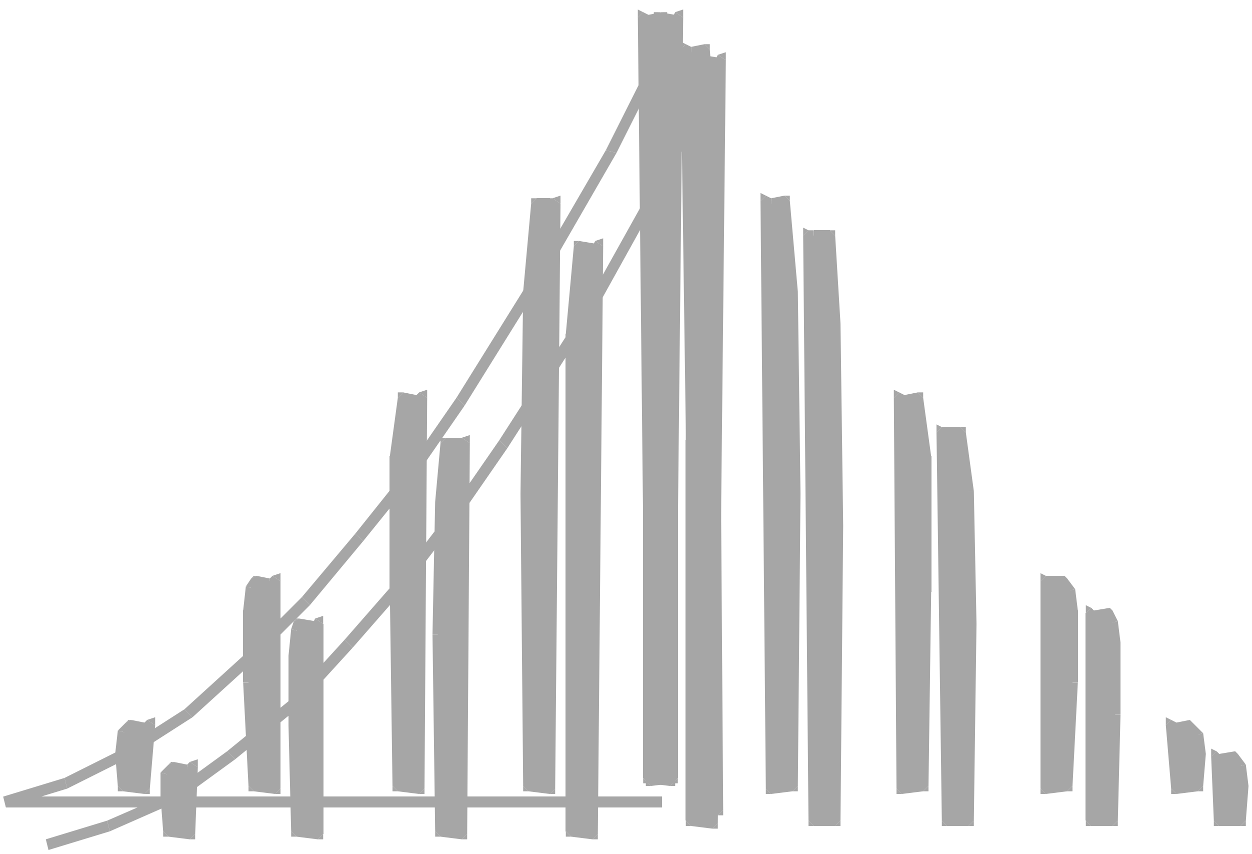

SAMPLER

BRIDGES
TO CLASSROOM MATHEMATICS

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BRIDGES

TO CLASSROOM MATHEMATICS

Bridges to Classroom Mathematics: a Staff-development Curriculum for Elementary School Teachers

Bridges to Classroom Mathematics is a professional development curriculum that supports teachers who are planning to implement innovative elementary mathematics curricula. The program is designed to emphasize the philosophy, purposes, and approaches of any curriculum aligned with the NCTM Standards.

Bridges Materials for Staff Developers
Bridges includes approximately 60 two-hour units that staff developers can combine flexibly to design workshops to meet local needs and resources.

Mathematical/Pedagogical units are designed for teachers who are using any standards-based curriculum or are preparing to make new curriculum choices. These units strengthen teachers' mathematical understanding and confidence, as well as model specific pedagogical techniques. **Curriculum-specific** Bridges units are designed for users of either *Everyday Mathematics* from the University of Chicago School Mathematics Project or *Investigations in Number, Data, and Space* ® from TERC.

Each Bridges unit includes a detailed Staff Developer's Guide, masters for handouts and overhead transparencies, and materials lists. Some units include a video.

Bridges also includes a Mathematics Handbook–a collection of essays about mathematical topics that support and enrich participants' mathematical content knowledge.

Creating Bridges Workshops
The Bridges materials are designed to build local school or district-level capacity. They provide complete presentation instructions and a comprehensive set of print materials that can be used successfully by local staff developers. Bridges workshops have been given successfully by presenters from a variety of backgrounds in a variety of situations.

- Schools or districts considering adopting standards-based elementary mathematics curricula can use Bridges materials to develop introductory hands-on workshops to increase staff understanding and appreciation of standards-based pedagogy.
- Bridges workshops can provide training and support to teacher-leaders and administrators, and direct in-service workshops for teachers in districts and schools that have already adopted a standards-based elementary mathematics curriculum.
- Bridges units have been effectively used to introduce pre-service teachers to standards-based elementary mathematics content and pedagogy.

The Bridges Sampler
This collection is a sample of the Bridges materials and includes:

- a complete list of all Bridges units
- thumbnail descriptions of the Mathematical/Pedagogical units
- a Mathematical/Pedagogical unit on Measurement
- an essay on Measurement from the Mathematics Handbook
- an *Everyday Mathematics* unit on procedures for multiplying and dividing
- an *Investigations in Number, Data, and Space* unit on number games

Please keep in mind that the Bridges materials, including video components, are still in development and in various stages of revision.

How to Order Bridges Materials
Draft versions of the printed Bridges Materials are available in configurations to meet different staff development needs. See the Bridges Materials Order Form that accompanies this document.

For more information about the Bridges curriculum visit our website at www.arccenter.comap.com. For assistance in selecting Bridges materials appropriate for your needs, call the ARC Center at (800) 772-6627, ext. 50 or email us at arccenter@comap.com.

Thank you for your interest in the Bridges to Classroom Mathematics project.

BRIDGES
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MATHEMATICS

BRIDGES PUBLICATION ORDER FORM

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DESCRIPTION	PRODUCT NO.	QTY	UNIT COST	TOTAL
Staff Developer's Guide: ***Mathematics for Elementary Teachers*** Includes: 19 Workshop Sessions / Mathematics Handbook / Video	4112		$111.00	
Staff Developer's Guide: ***Investigations in Number, Data, and Space*** Includes: 16 Workshop Sessions / 50 Unit Guides / Video	Print 4114		$165.00	
	CD-ROM 4132		$99.00	
Additional Mathematics Handbooks	4115		$28.00	
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***Investigations* Package** Includes: Staff Developer's Guide: *Mathematics for Elementary Teachers* Staff Developer's Guide: *Investigations*	Print 4117		$255.00	
	CD-ROM 4134		$149.00	
***Everyday Mathematics* Package** Includes: Staff Developer's Guide: *Mathematics for Elementary Teachers* Staff Developer's Guide: *Everyday Mathematics*	CD-ROM 4135		$105.00	

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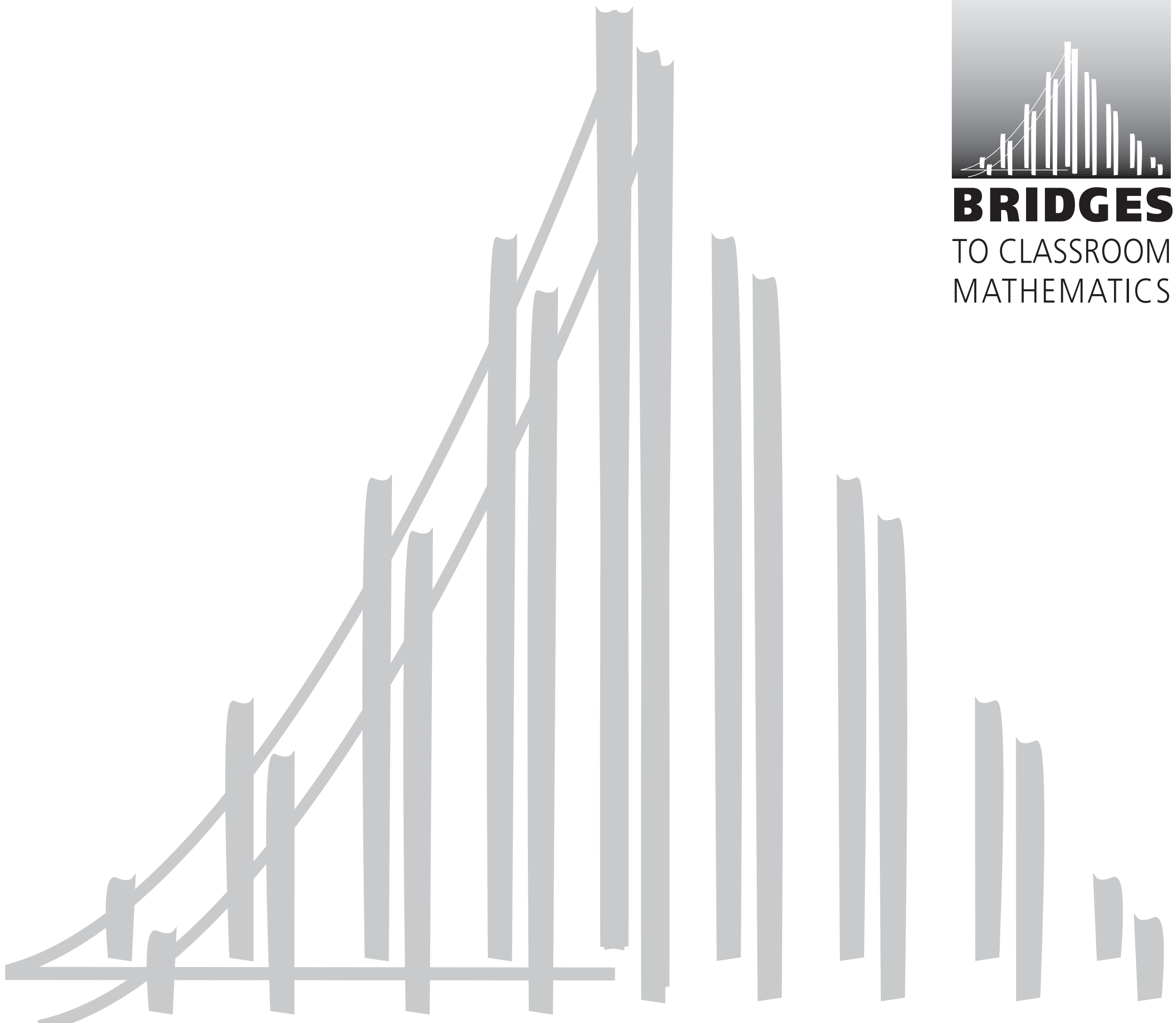
MATHEMATICS



Unit Lists and Thumbnail Descriptions

Generic Mathematical/Pedagogical Units

- Inventing Computational Procedures
- Number and Counting
- Arrays and Factors
- Combining and Comparing Fractions
- Fractions, Decimals, and Percents
- Real Numbers
- Introduction to Ratio
- Processes of Data Analysis
- Exploring and Representing Complex Data
- Probability
- Exploring Polygons
- Solid Figures
- Angles and Polygons
- Area Explorations
- Measurement
- Patterns and Functions
- Number Theory
- Conjecture and Proof
- Assessment

Everyday Mathematics Bridges Units

- Addition and Subtraction in *Everyday Mathematics*
- Multiplication and Division Concepts in *Everyday Mathematics*
- Procedures for Multiplying and Dividing in *Everyday Mathematics*
- Fractions in *Everyday Mathematics*
- Data in *Everyday Mathematics*
- Probability in *Everyday Mathematics*
- Geometry in *Everyday Mathematics*
- Patterns & Functions in *Everyday Mathematics*
- Algebra in *Everyday Mathematics*
- Assessment in *Everyday Mathematics* I
- Assessment in *Everyday Mathematics* II

Investigations in Number, Data and Space Bridges Units

- Introduction to *Investigations*: Overview and Structure of a Unit
- The Role of Games in *Investigations*
- Number in *Investigations*
- Geometry in *Investigations*
- Fractions in *Investigations*
- Data Analysis in *Investigations*
- The Mathematics of Change in *Investigations*
- Assessment (with *Investigations* information)

Investigations Unit Guides are available for:

- Introductory Units K–2
- Landmark Units 3–5
- Number Units K–5
- Fractions 3–5
- Geometry, Data, and Mathematics of Change K–5

Bridges Units Currently in Development

- Issues in Reform Math Grades K–1
- Issues in Reform Math Grades 2–3
- Issues in Reform Math Grades 4–5

Bridges Mathematics Handbook (essays)

- Number and Counting
- Operations
- Rational Numbers
- Functions
- Data Analysis
- Probability
- Geometry
- Measurement

Thumbnails: Mathematical/Pedagogical Units

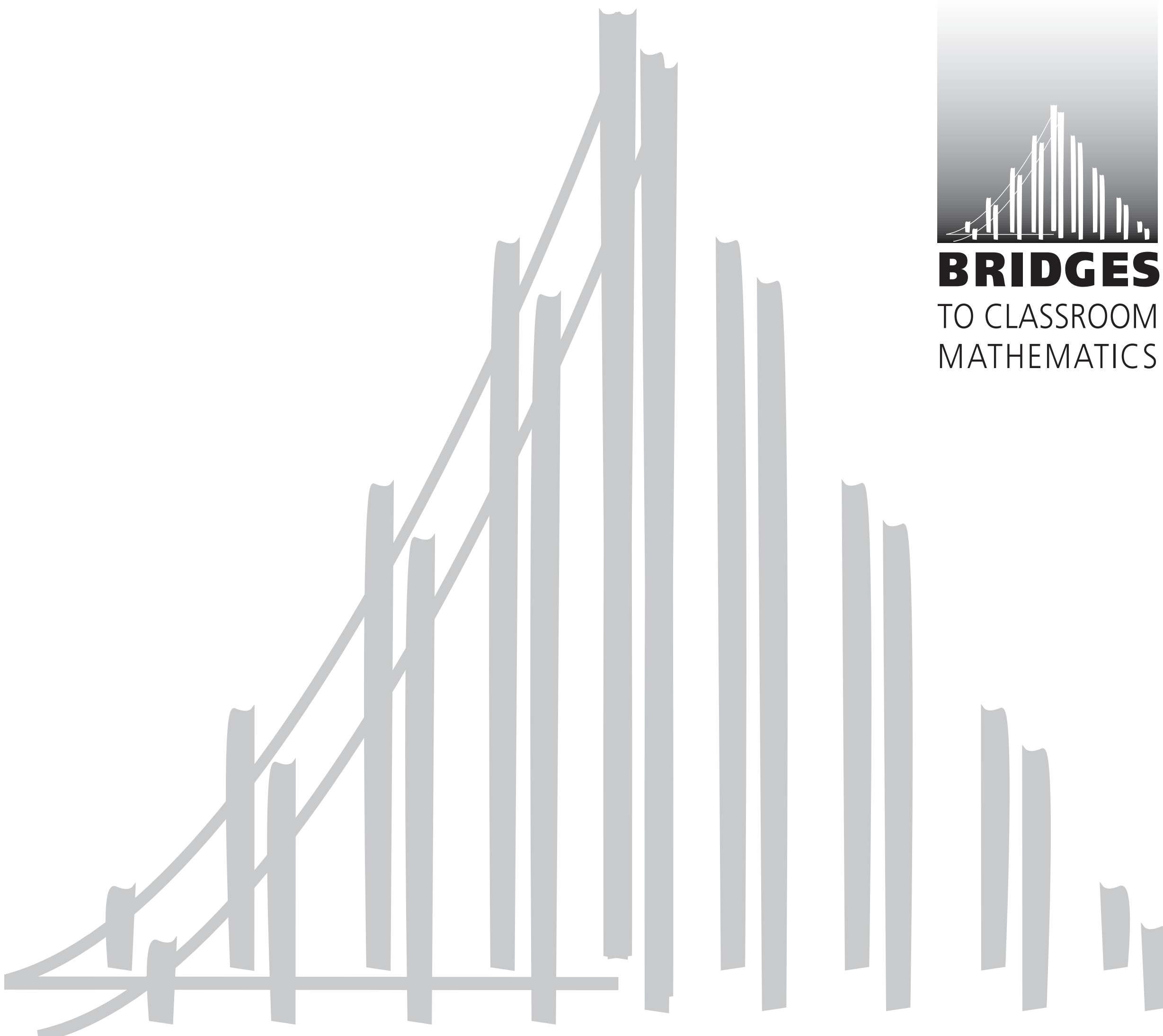
The "generic" units are intended to strengthen teachers' mathematical understanding and confidence, as well as to model pedagogical techniques. Each unit takes about two hours in a workshop setting.

Inventing Computational Procedures	Participants explore the basic number and operations knowledge that is used in everyday computation and develop and use mental and written computation strategies. Using video, participants observe and evaluate children's problem-solving strategies.
Number and Counting	Participants investigate the variety of uses of number; distinguish between number and different notations; explore the connections between counting, place value, and basic operations; and identify patterns in the natural numbers, using number grids, to review basic number theory.
Arrays and Factors	Participants explore multiplication and division by combining and partitioning equal groups. Array models are used to find factors and relate multiplication and division. Learning and use of basic facts for efficiency and the value of teaching multiplication and division simultaneously are emphasized.
Combining and Comparing Fractions	Participants develop a repertoire of ways to compare fractions, beyond finding common denominators, including making wholes from fractional parts, describing fraction combinations with addition sentences, connecting fraction names to parts of groups, and finding familiar fractions that are close in size to unfamiliar fractions. A measurement model is used to compare fractions.
Fractions, Decimals, and Percents	Participants observe the relationships among whole and fractional parts using a variety of part/whole models. They explore the idea of equivalence and work with converting fraction, decimal, and percent notations.
Real Numbers	Participants explore the family of real numbers, including the natural (counting) numbers, negative, rational, and irrational numbers. The Pythagorean theorem is reviewed. Participants learn about the differences between rational and irrational numbers.
Introduction to Ratio	Participants review the concept of ratios as comparisons of two quantities and explore the naturally occurring constant ratio of circumference/diameter. They invent procedures for solving familiar problems involving ratios, and discuss applications of ratios.
Processes of Data Analysis	Participants gather, observe, and describe data in terms of individual features and overall distribution. Line plots and other representations are constructed. Statistical concepts, fractional parts, and percentages are used to compare data sets. Participants consider the development of children's understanding of data across the elementary grade levels.

Exploring and Representing Complex Data	Participants organize the analysis of a complex data set. Accuracy is discussed and defined. Different representations are compared for clarity and complexity. A variety of data formats are used.
Probability	Participants consider two different methods of formulating probabilities: experimental and theoretical. Systematic listings, tree diagrams, and grids are used to generate a set of all possible outcomes. Participants compute the expected number of events in a sample from the probability of the event.
Exploring Polygons	Participants review the defining properties of polygons and derive a working definition of polygon by comparing characteristics of examples and counterexamples. The relationship between line symmetry and reflections is explored; polygons are classified by lines of symmetry. Participants investigate relationships among lengths of triangle sides.
Solid Figures	Participants review and construct geometric solids, investigate two-dimensional representations of solids, and reflect on the substantial, informal experience of children with regard to three-dimensional figures. Construction of polyhedra leads to derivation of Euler's formula ($v + f = e + 2$).
Angles in Polygons	Participants classify shapes according to attributes. They use indirect methods to find polygon angles, and develop a generalization about the sums of the angle measures in a regular polygon.
Area Explorations	Participants explore methods for finding areas of figures and recognize that areas can be compared without using numbers. They recognize relationships among different shapes with the same area, establish properties of similar figures, and explore relationships between side-lengths and areas of similar figures.
Measurement	Participants estimate and measure various lengths, investigate the relationship between area and perimeter, and work with objects of various weights in order to gain an intuitive understanding of approximate weight. The concept that a square minimizes perimeter and maximizes area is considered. Measurement error is discussed.
Assessment	Participants explore an approach to assessment emphasized in reform curricula: observing and talking with students as they work on assessment. They consider strengths and challenges of this and other approaches to assessment..
Patterns & Functions	Participants identify mathematical processes involved in pattern tasks and develop an understanding of patterns and functions as rule-based relationships. Representations of functions are discussed.

Number Theory	Participants generate special sequences of numbers: primes, powers of two, squares, and triangle numbers. They review ways of factoring numbers and investigate the number sequences to find relationships within and among them.
Conjecture & Proof	Participants make general arguments about arithmetic and distinguish between proof and evidence from cases. The relevance of conjecture and proof to school mathematics is considered.

S A M P L E R



BRIDGES
TO CLASSROOM
MATHEMATICS

A Mathematics Handbook Essay

MEASUREMENT

All measurement involves comparison. Ideas like "tall," "heavy," and "hot," have meaning only when related to something else having the same attribute—tall compared to what? Heavy compared to what? Hot compared to what?

Throughout history, humans have developed increasingly more precise methods for comparing attributes. At first, direct comparison probably served well enough: this person is taller than that person; this stone is heavier than that stone; this fire is hotter than that fire. Later, indirect comparisons were made possible by using some common objects as "units." For example, lengths of objects were compared to various body parts. A "foot" was the length of a man's foot. Since about 12 thumb widths equal a foot's length and the Latin word for "one twelfth" is uncia, today we have 12 inches in a foot. An "ell," often used for measuring fabric, was the distance from the nose of a man to the end of his fingertips. A thousand paces of a Roman soldier was a "mile." (Mille is the Latin word for "thousand.") Temperature scales have been devised based on the freezing and boiling points of water and on the temperature of the human body. An "acre" was the amount of land a horse could plow in one day. As you can imagine, these "units" allows for wide variation, so it became increasingly necessary to standardize units of measure.

Systems of Measurement

Various cultures developed their own systems of standardized measurements, but the two we use most today are the English or U. S. customary system, and the metric system. The English system was developed around the thirteenth century. For example, the ell mentioned above was "standardized" as the measure of the distance from the king's nose to the end of his arm and is known as a "yard." Over time this system became standardized, meaning reproducible, to the point where there was an "official foot" in Great Britain that was a metal rod one foot long. One-twelfth of this foot was an inch, three of them a yard, 5280 of them a mile. The units for weight, capacity, time, and temperature were standardized as well.

One of the problems with the English system is that its units for length, weight, capacity, time, and temperature are unrelated to each other. Another is that the relationships between units for the same attribute are cumbersome. For example, a foot is 1/3 of a yard, but an inch is 1/12 of a foot. An ounce of liquid capacity is 1/32 of a quart, but an ounce of dry weight is 1/16 of a pound—unless, for example, gold is being measured, in which case there are 12 "Troy" ounces in a pound! And so, although a Troy ounce is slightly heavier than a regular ("Avoirdupois") ounce, a pound of gold actually weighs less than a pound of feathers!

The Metric System

At the end of the eighteenth century, the metric system was developed in France. The intent was to devise a system of standard measures that were interrelated and easy to convert from one to the other. The meter was the basic unit from which others were derived. The meter was originally defined as one ten millionth of the distance from the North Pole to the equator along the meridian through Paris.

This geographic basis, however, proved inconvenient in practice, and eventually an International Prototype Metre was created. The prototype meter was a rod of platinum and iridium that served as the worldwide "official" meter and was kept at the International Bureau of Weights and Measures in Sèvres, France. Copies of the prototype meter and the prototype kilogram were also kept in Washington, D. C., and other world capitals.

Eventually, however, the requirements of modern science demanded a definition of a meter more exact than the length of a particular metal bar in France. Today, a meter is defined as 1,650,763.73 wavelengths of the orange-red light of krypton 86. This definition is much more precise than the old definition, and has the additional advantage that anyone with the appropriate equipment can reproduce a true meter without having to travel to France to use the rod. (Interestingly, the kilogram is still defined in terms of the standard masses at the International Bureau of Weights and Measures. So if you want to check whether your kilogram mass is really correct, you have to go to France!)

From the meter, other units of length were defined using a system based on powers of 10: A decimeter is 1/10 meter, a centimeter is 1/10 decimeter and 1/100 meter, and so on. Additionally, in the metric system, units of mass, volume, and capacity are related to units of length: At standard conditions (4°C and 1 atmosphere of pressure), the mass of 1000 cubic centimeters of water is 1 kilogram, which is the same as saying that 1 cubic centimeter of water weighs 1 gram. A liter is 1000 cubic centimeters, the volume of a cube 10 decimeters on a side. Since the same prefixes (kilo-, deci-, etc.) are used with all measurements, whether for length, mass, or capacity, just one set of relationships has to be learned. All these features make the metric system much easier to learn and use than other systems. For example, if you know the volume of a water-based liquid, you can easily estimate its weight. And the decimal relationships between units make conversions as easy as multiplying or dividing by powers of 10.

The Measurement Process

Measuring involves determining what attribute of an object is to be measured and comparing the object to a unit with the same attribute. To measure the length of something, for example, we choose a unit that has length, like a handspan or an inch. Measuring the object's length then involves determining how many of these length units are equivalent to the object's length. We can use either standard or nonstandard units to measure things. For estimates and some teaching purposes, we often use nonstandard units like pace lengths, hand spans, or the length of a pencil. For many everyday purposes, nonstandard units are sufficient.

For more uniform results, however, standard units of measure are essential. This generally involves the use of measurement instruments that have been calibrated to varying degrees of precision. The precision of a measurement is related to the unit of measure—the smaller the unit of measure, the greater the precision. A measurement to the nearest 1/4 inch is more precise than one to the nearest 1/2 inch. How precise a measurement we need will determine the instrument we use. If we are measuring the distance between two cities we need a less precise measurement than if we are measuring a window for curtains.

Measurements Are Approximate

All measurements are approximations. The main reason for this is that no matter how small a unit of measure we use, there is always a smaller one that could, in theory, be chosen, and the smaller unit could give a different, more precise answer than the larger unit. We might, for instance, measure time to the nearest second, but if our instruments would allow, time could be measured to the nearest tenth, hundredth, thousandth, or ever smaller parts of a second. This also apples to measures of length, area, and every other attribute we can measure. Using an ordinary ruler, for example, we can measure the length of a pencil to the nearest millimeter, but with better instruments we could measure to the nearest micron or to even greater accuracy.

Yet at some point in our quest for ever greater precision we cannot even tell where the ends of the pencil are to measure them! This is so because the atoms that form the pencil are surrounded by "clouds" of electrons that do not have sharp boundaries. So at a very great level of precision we cannot even tell exactly where the ends of the pencil are in order to measure the distance between them. This is another reason that measurements are approximate: The objects we are measuring are sometimes not completely well-defined. As we breathe, for example, our weight changes very slightly, but by an amount that we could detect if we had precise enough measuring tools.

The approximate nature of all measures distinguishes counts from measures. It is possible to count all the people in a room and get an exact answer because there are no "half people"—either there is a person to be counted or there isn’t, there’s nothing in between. But if we were to measure the height of one of these people, we could not be as sure of our answer. This is so, first, because no matter the unit we choose, there will always be a smaller unit that would give a more precise measurement of that person’s height, and second, because the person’s height is not completely stable and well defined: she might slouch or stand up straight, or move her head slightly, causing her height to change slightly.

Another way to say that measurements are always approximate is to say that there is always some error when a measurement is made. If, for example, we measure a pencil to the nearest centimeter and find that it is 11 cm, the pencil might really be 11.4 cm long, or 10.6 cm, or any other length between 10.5 cm and 11.5 cm. The difference between the number we read from our ruler and the "true length" of the pencil is called the measurement error.

Measurement error is not the same as a blunder or a mistake. Even if we are very careful and use the best measuring instruments we can find, there will always be some measurement error. The real issue with measurement error is controlling it, both trying to keep it as small as possible and trying not to let it interfere with our conclusions. One way to control measurement error is to make the same measurement several times and then to average the measurements. Another way is to graph the measurements so that any underlying patterns are easier to see.

Commonly Measured Attributes

Length

The **length** of an object is a measure of the distance between two points at the ends of the object. It is a measure of one dimension of an object. **Perimeter**, the distance around the boundary of a region, is also a measure of length. The perimeter of a circle is called the circumference of the circle.

Area

Area is the amount of surface in a two-dimensional region. Area is harder than length to measure directly because we have no convenient tool like a ruler for measuring area. One way to measure area directly is to trace the region to be measured on a grid. Then the area of the shape is the number of unit squares it covers. If the shape is irregular, this number must be estimated by joining together pieces of units. Figure 1, for example, shows a shape with an area of about 13 square centimeters.

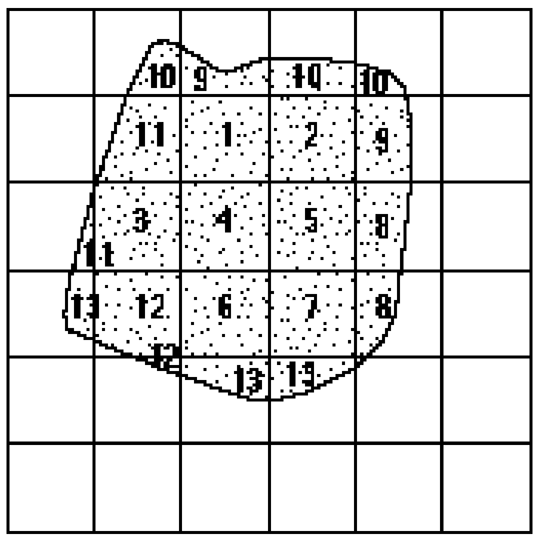


Figure 1. A irregular shape with area about 13 sq. cm.

The areas of more regular shapes can be easier to find. If a rectangle, for example, has length and width that are whole numbers of units, then the number of square units needed to cover it is a whole number too. Figure 2 shows that a 7 by 3 rectangle has an area of 21 square units. If the grid squares are 1 inch on each side, then the rectangle's area would be 21 square inches; if the grid squares are 1 meter on each side, then the rectangle's area would be 21 square meters.

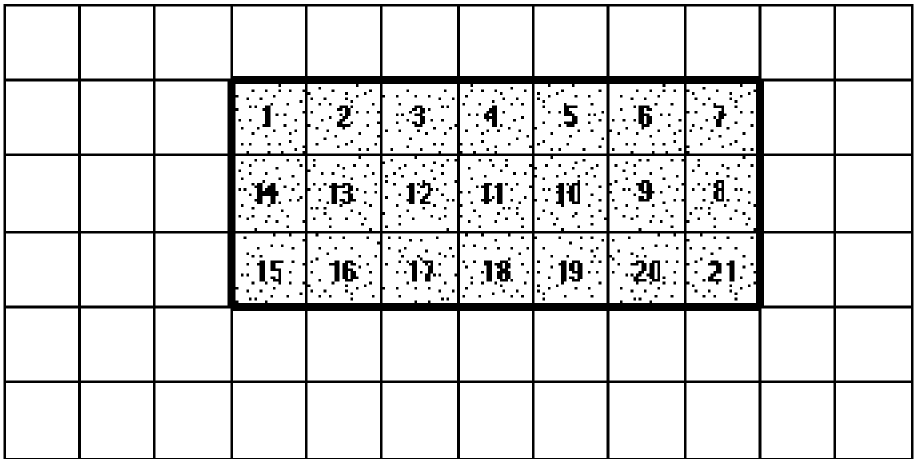


Figure 2. A rectangle with area 21 sq. units.

We know, of course, that it's not necessary to count to find the area of the rectangle in Figure 2. It's much faster just to multiply the number of squares in each row (the width) by the number of rows (the length). As we know, this "short cut"—measuring the length and the width and multiplying them together—works for any rectangle, even those with side lengths that are not whole numbers.

Another way to visualize this process is to think of covering or tiling a rectangle (see Figure 3) with two-dimensional units like squares, and then counting the number of these units. This model of area—covering or tiling a shape with two-dimensional units—is useful with young children because it is easier to understand and it allows the students to use square inch tiles, for example, to cover a rectangle and then to count to find the area.

Yet another way to visualize area is not as a count of tiling units, but as a length sweeping out part of a plane. For example, if we have a rectangle 5 cm by 8 cm, we can imagine a line segment 5 cm long "sweeping or painting" over a distance that is 8 cm long (see Figure 4).

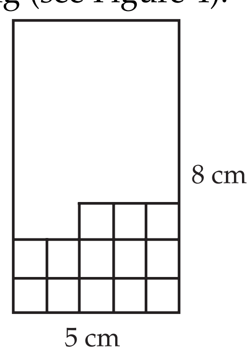


Figure 3. Covering or tiling a space.

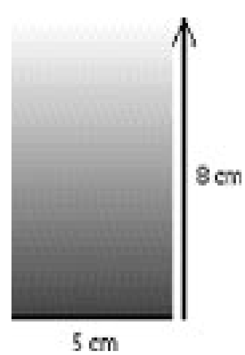


Figure 4. Sweeping or painting a space.

Volume

Volume is the amount of space an object occupies. Units for volume are cubic—cubic centimeters, cubic inches, cubic feet, cubic meters, etc. A cubic foot, for example, is a cube with a length of one foot, a width of one foot, and a height of one foot.

Volume can be measured in several ways. If we consider a box—a rectangular prism—with dimensions 9 inches by 4 inches by 3 inches, we can think of filling it with inch cubes and counting the cubes to find the volume of the box, 108 cubic inches (see **Figure 5**).

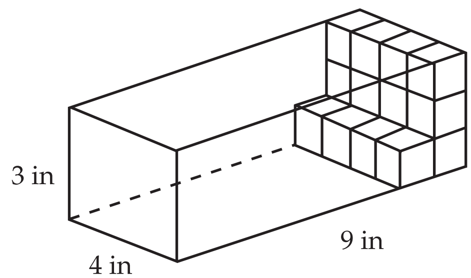


Figure 5. Filling a box with cubes.

As with area, there is a short cut to counting all these unit cubes. The area of the base of the box, 4 in. x 9 in. = 36 sq. in., tells us the number of cubes in each layer, and the height of the box, 3 in., tells us the number of layers. So the volume of the box is the area of the base times the height. This short cut, like the short cut for the area of a rectangle, works even when the lengths of the sides are not whole numbers. It even works for other prisms and for cylinders, although in these cases finding the area of the base is not quite as easy as for a simple box.

Another way to visualize volume is similar to the "sweeping" out of an area. Imagine the base of the box, a 9-inch by 4-inch rectangle. Then imagine "sweeping" this rectangle through the 3-inch height filling the space (see **Figure 6**).

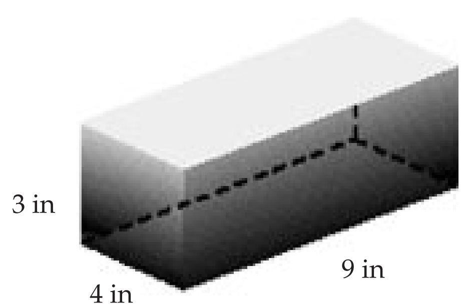


Figure 6. Sweeping out a space.

Another way to measure volume requires a bit of clever thinking. Archimedes, the great mathematician and scientist who lived in Sicily in the third century B.C., was the first to discover this method. Hieron, the king of Syracuse, suspected a goldsmith of mixing silver with the gold he was given to make a crown. Hieron asked Archimedes to expose the theft, but to do so he had to find the volume of the crown. Finding the volume of an irregular object like a crown, however, is much harder than finding the volume of a box. Archimedes thought about the problem for days until at last as he was getting into his bath he noticed the water level rise. At once Archimedes had the idea he needed and leaped out of his bath and ran naked through the streets crying, "Eureka! I have found it!"

What Archimedes had discovered is that the volume of an object can be measured by immersing the object in liquid and noticing how much the surface of the liquid rises. For example, suppose you have a graduated cylinder with 70 cubic centimeters of water in it and a marble whose volume you want to know. If the water level rises to 75 cubic centimeters when the marble is immersed, then you would know that the volume of the marble is 75 cc – 70 cc = 5 cc. See Figure 7. The discovery of this technique, sometimes called finding volume by displacement, is what drove Archimedes on his famous run through the streets of Syracuse so long ago.

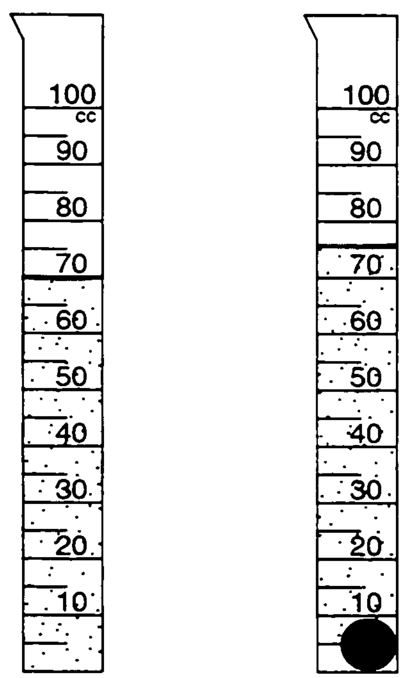


Figure 7. Finding the volume of a marble by displacement.

Each of these—counting cubes, area of the base times the height, displacement—is a legitimate approach to volume. With young children, we prefer the method that uses the idea of small cubes to measure the volume of a larger object. By third or fourth grade, though, students can explore the area-of-the-base-times-the-height method by gradually filling containers with water or sand. At about the same age they can also find the volume of small objects by displacement.

Weight and Mass

Mass is the amount of matter in an object. Weight is how hard gravity pulls on an object. If you took a trip around the solar system and weighed yourself on each planet, you would find your weight changing dramatically depending on the size of the planet. Big planets would make you weigh more because their gravitational pull is strong. On small planets you would weigh less because the planet wouldn't exert as much pull. Your mass, though, remains the same no matter which planet you're on or even if you're in interplanetary space. The amount of matter in your body is not affected by gravity.

We note this distinction for you, but since we and the objects we weigh are usually firmly planted on the earth, it is of little practical consequence. With young children, it is neither necessary nor appropriate to make any distinction between mass and weight.

Weight can be measured using a variety of scale types. In most simple scales the stretch of a spring or the bending of a rod is used to gauge the force of gravity on the object being weighed. To measure mass, a balance can be used to compare an object's mass to a standard mass set. Notice that if you took a spring scale to the moon, the force of gravity on the object being weighed would be less, so the spring would stretch less, and the measured weight would be less. If you use a balance and standard masses to find an object's mass, on the other hand, then it doesn't matter if you are on the earth or the moon: the weaker pull of gravity on the moon affects both the object being measured and the standard masses equally, so the measured mass would be the same no matter which planet you're on. (In "weightless" conditions, however, a balance cannot be used to measure mass because both the object and the standard masses would just float around. To find the mass of an astronaut in orbit is hard.)

Angular Measure

Angular measures are used to quantify turns or rotations. The most common unit for measuring angles is the degree. A **degree** is 1/360 of a full turn; see **Figure 8**.

Figure 8. An angle measuring one degree (1°).

For most practical situation, measurement to the nearest degree is accurate enough, but for applications like astronomy or surveying that require more precise measurements, each degree can be divided into 60 minutes (60') and each minute into 60 seconds (60'').

The degree is the most common unit for angular measure, but there are several others. A revolution is one complete turn; it is used, for example, in measuring the speed of engines. A grad, a unit used in engineering, is 1/400 of a full turn. A radian, used in higher mathematics, is the measure of the angle that cuts off an arc of a circle equal in length to the radius of the circle. A radian is about 57°; see Figure 9. Many calculators can work in degrees, radians, or grad; you may have seen a calculator's display shift from DEG to RAD to GRAD.

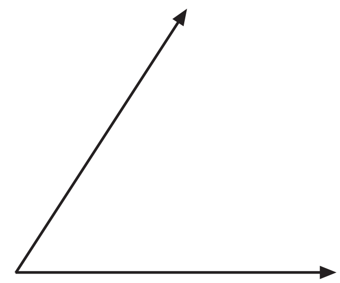


Figure 9. An angle with measure one radian.

Capacity

Sometimes we need to know amounts of things like liquids or small grains of sand or sugar, things that take the shapes of their containers, things that can't be measured by their lengths, widths, and heights to get a volume. In such situations, we use measures of capacity. In the U. S. customary system we use containers marked in cups, quarts, gallons, etc.; in the metric system we mainly use liters and milliliters. A milliliter is 1 cubic centimeter; a liter is 1000 cubic centimeters.

Temperature

Temperature scales are developed by choosing a cold point and a hot point and then marking equal gradations between the two (see Figure 10). Each of these gradations is called a "degree." There are two main temperature scales that are used in everyday life, the Fahrenheit scale and the Celsius or centigrade scale. The Fahrenheit scale uses the freezing (32°F) and boiling points of water (212°F) as fixed points. The Celsius scale is part of the metric system and also uses the freezing (0°C) and boiling points of water (100°C) as fixed points. To convert measurements between Fahrenheit and centigrade, you can use either C = 59 (F – 32) or F = 95 C + 32, depending on which direction you want to do the conversion.

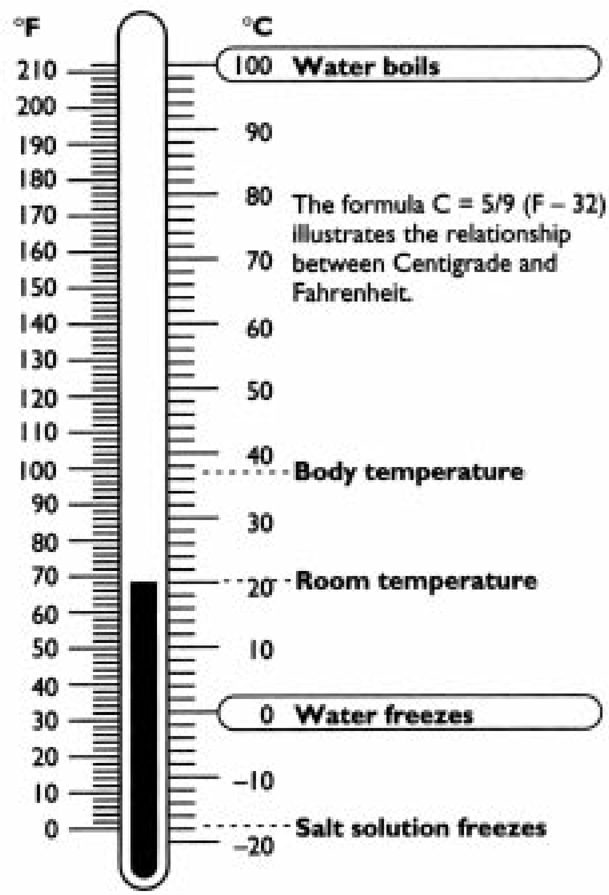


Figure 10. Thermometer.

Note that 0° on the Fahrenheit and Celsius scales does not indicate the absence of heat, in the way that 0 pounds or 0 inches indicate an absence of weight or length. Zero on these temperature scales is not a true zero; it is just an convenient anchor point for establishing a relative scale.

Another important temperature scale is the Kelvin scale. A degree Kelvin is the same size as a degree centigrade, but the zero point on the Kelvin scale is absolute zero, the temperature at which all molecular motion stops. Absolute zero is about –273°C, a temperature so cold that air would be frozen solid. Water freezes at about 273K; room temperature is about 293K. The Kelvin scale is used almost exclusively in the sciences. (The "degree" symbol is not used in Kelvin notation.)

Time

Our standard units for measuring time have a long history. The ancient Egyptians used a 24-hour day divided into two 12-hour periods, day and night. Each of the hours was named for one of the seven largest celestial bodies visible to them—the five visible planets plus the sun and the moon. Through a system of cycling those seven names through successive 24-hour periods, they arrived at a week of seven days.

The ancient Babylonians, who used a base of 60 in their commerce and trade (probably because 60 can be divided evenly by many numbers), gave us the 60-minute hour and 60-second minute. (The Babylonians also gave us the system of dividing the circle into 360 degrees and the subdivisions of the degree into minutes and seconds.)

In a certain sense, temperature readings and clock time are not measures in the same way that lengths are, for example. A reason for this is that measurements are combinable—statements like 5 cm + 6 cm = 11 cm make sense—whereas readings from a thermometer or clock are not. We cannot, for example, take temperature readings of 5°C and 6°C, add them together and get 11°C. Nor can we add 5 o'clock and 6 o'clock and get 11 o'clock. Clock time and temperatures are more like latitude and longitude: they tell us where we are in some frame of reference.

We can, however, say that the temperature rose 5°, then rose another 6° for a total increase of 11°. Also, we can say that one occurrence lasted 5 minutes and a second occurrence lasted 6 minutes, giving a total of 11 minutes for both. In these cases, elapsed time and change in temperature, we are dealing with true measures.

Direct and Indirect Measurement

Most measurements are made by comparing an object to be measured directly with some measuring tool. A pencil can be measured with a ruler. The circumference of a tree can be measured by passing a tape measure around it. A handful of spices can be weighed with a small scale. These are called direct measurements.

Sometimes, however, direct measurement is impossible. A tree may be too tall for its height to be measured with a ruler or tape measure. A fire may be too hot to measure its temperature with a thermometer. In cases like these, when direct measurement is impossible or impractical, there are often indirect ways to make the desired measurement. The height of a tall tree, for example, can be found by comparing its shadow with the shadow of an object with known height. (See Figure 11.) The ratio of the shadow lengths equals the ratio of the heights:

$$\frac{\text{tree height}}{\text{stick height}} = \frac{\text{tree shadow}}{\text{stick shadow}}$$

Since three of these quantities can be measured directly—the stick's height and both the shadows—the fourth one can be found too. This is an example of **indirect measurement**.

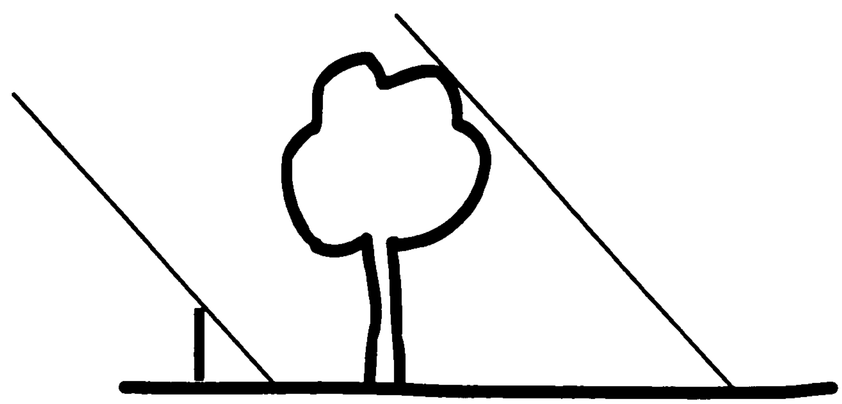


Figure 11. Indirect measurement of a tree's height.

Estimation and Rules of Thumb

Skill at estimation takes a long time to develop. The people who are best at estimating a certain kind of measurement are professionals in a field that requires that sort of knowledge. For example, carpet layers can estimate the area of a room quickly and accurately. Hog farmers can estimate the weight of a pig.

A high level of estimation skill takes years to develop, but an improved level of skill can often be achieved relatively easily. Developing "personal benchmarks"—the width of an adult's thumb, for example, is about an inch—is a big help. Simple techniques like "chunking"—estimating a ceiling's height, for example, by how "people" high it is—can also help.

Also useful are rules of thumb, measurement estimates based on intuition and experience. Below are some examples. Do you have any others to add to the list?

1. To make sure that a sock you buy is the right size, wrap the foot of the sock around your fist. If the heel of the sock meets the toe, the sock will fit.

2. To estimate the degrees Fahrenheit outdoors, count the number of chirps a cricket makes in 15 seconds and add 37.

3. The length of your foot will fit between your elbow and your wrist.

4. To tell about how many miles away you are from a thunderstorm, count the seconds between the lightning and the thunder and divide by 5.

5. "A pint's a pound the world around."

6. One squeezed orange makes about half a cup of juice.

7. A quick way to estimate the conversion from Centigrade to Fahrenheit temperatures is to double the Centigrade temperature and add 30.

Measurement and Young Children

As with many other areas of mathematics, young children come to school with quite a bit of informal knowledge about measurement. They have extensive experience making direct comparisons: this person is taller than that person, this glass has more milk than that glass, this stone is heavier than that stone. They also generally know how old they are, that people are "something feet tall," that time is measured in hours and distance in miles, and so on. In other words, by the time they come to school, children have had significant exposure both to the attributes we measure and to the names of the units we use when measuring those attributes.

Contemporary elementary school mathematics curricula seek to build on this informal knowledge by giving young children lots of first-hand experiences in measuring. Children use a variety of instruments to measure the attributes of length, area, volume, weight, capacity, time, and temperature. These measurement activities are embedded in real contexts, often data collection, in a way that allows students to see the purposes for taking measurements and to make sensible judgments about their results.

Units

To understand the measurement process, students must understand the role of units: when measuring they are comparing an object to an inch, a centimeter, etc., and counting how many of those units are needed to equal the object. It is important that students understand the need to report both the number of units and the name of the units. One reason for using non-standard units with young children is to emphasize the importance of specifying the unit: If a child says the length is 14, you should ask, "Fourteen what? 14 toothpicks? 14 finger widths? 14 paper clips?" Using non-standard units highlights the importance of the unit in the measurement process. With lots of first-hand experience in measuring, children build strong understandings of which units are appropriate for a given purpose.

Students also should realize that all units in a given measurement must be the same size. If a child use paper clips to measure a distance, but some of the clips are large and others are small, then the measurement will be flawed. If the child uses pencils as the unit, but some are stubby and others are long, then again the measurement will be faulty.

A related point that is difficult for many children to understand is that as the unit gets larger, fewer of them are needed to equal the object being measured. A length that measures 60 with a small unit (inches), for example, can be only 5 with a larger unit (feet). This "inverse" relationship between the size of the unit and the number required is confusing for many children.

Using Instruments
Our experience is that if students are given appropriate guidance and lots of practice, they learn to use measuring instruments fairly rapidly. But still there are certain difficulties that are fairly common. One problem many beginners have is that they fail to line up the object they're measuring with the zero mark on the ruler, probably because their attention is focused on the other end of the object. To find the correct length of the rod in Figure 12 is a hard problem, even for many older children.

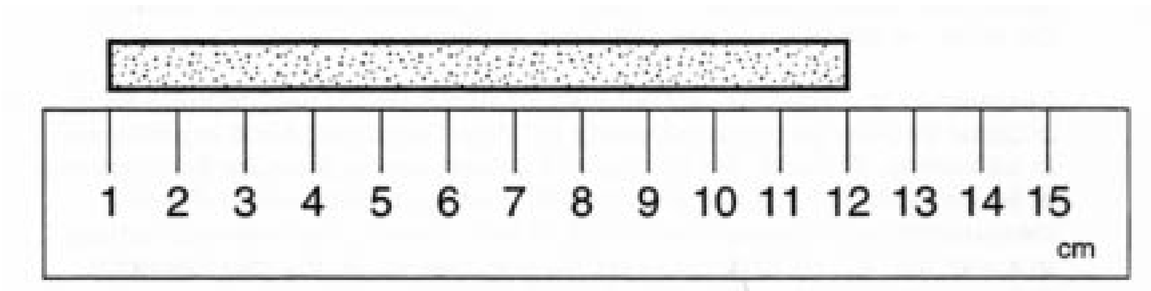


Figure 12. A measurement problem.

Standard instruments usually have numbers and markings between the numbers. Young children who are not familiar with fractions may have difficulty reading measurements that show fractions of units, e.g., 1/4 inch. We find that most young students ignore these markings and read the nearest number on the scale. This is fine, or the students can learn to read such measurements as "Eight inches and a little bit more."

Another error that children sometimes make is to count the marks rather than the spaces on a ruler. Thus, for example, a student who counts the marks next to the rod in Figure 7 would find the length to be 12 cm, whereas a student who counts spaces finds the correct length, 11 cm.

U. S. Customary vs. Metric
For most everyday purposes, this country still uses the customary system of feet, gallons, degrees Fahrenheit, and so on. Most of the rest of the world and all of the sciences, however, use the metric system. So elementary school mathematics in this country must deal with two systems. Fortunately, this appears not to be problematic for students. It is somewhat analogous to learning two languages—if the learners are young, they have no problem switching from one to the other as they find appropriate.

"About"
That all measurements are approximate is not much of a problem in daily life. Most measurements in real life need not be very exact. We only need to know "about"

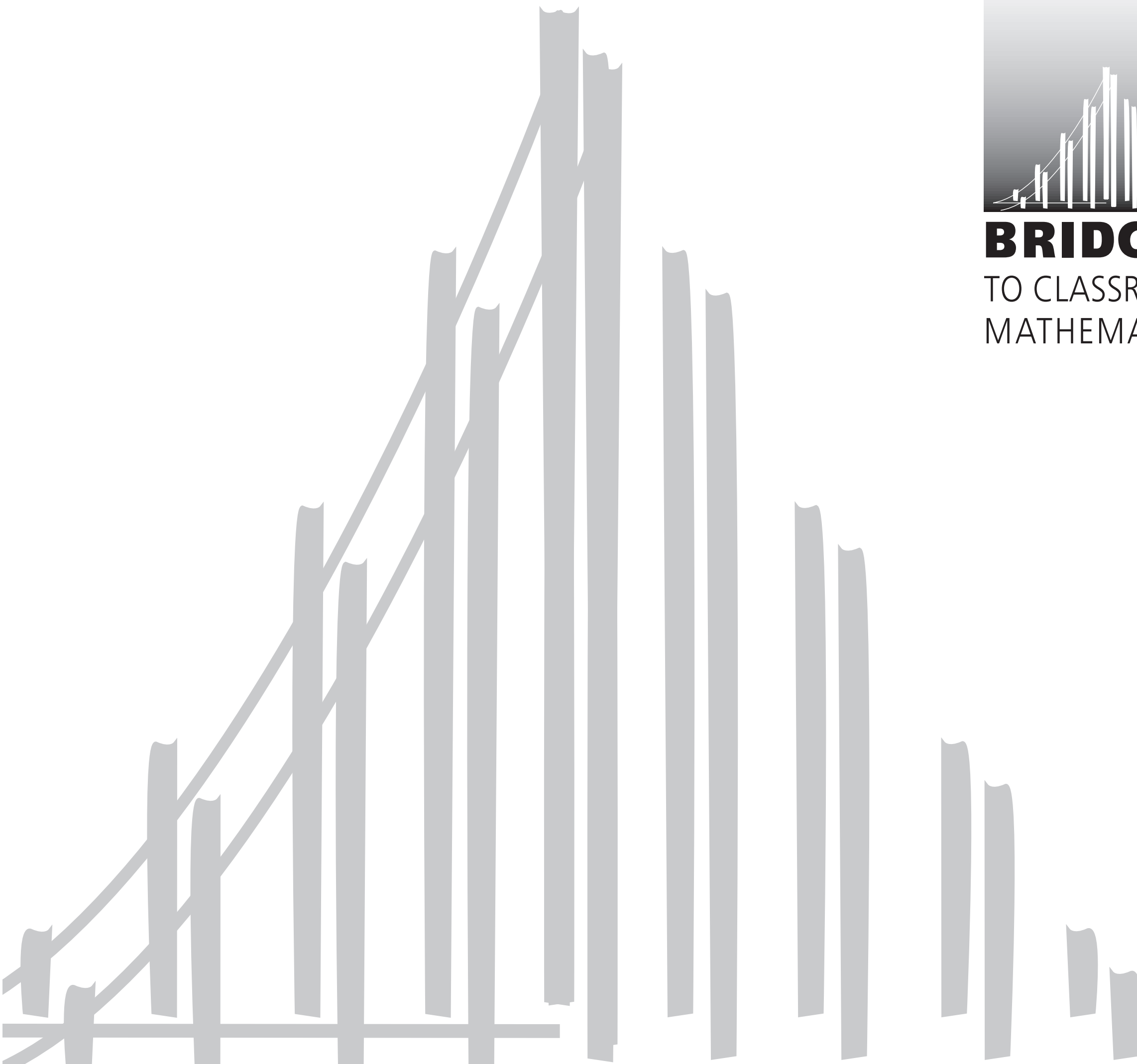
how far or how heavy. Students should have experiences that help them arrive at good estimates. Taking lots of measurements is the best way to get a feel for what a pound is, for example. Students can also be helped to make good estimates by finding handy references and rules of thumb. Students can learn to use parts of their bodies to estimate distances. Teachers should also encourage students to estimate measurements before taking more accurate readings. This development of "measure sense" is a major purpose of measurement activities in the early grades.

Conclusion

Measurement is more than simply a topic in the mathematics curriculum. Measurement connects mathematics to science, to the social sciences, to architecture and art, and to everyday life—it is one of the most common and useful parts of mathematics. Moreover, when children measure, they deal with concepts embodied in objects and represented in pictures, numbers, and formulas. As they measure, children can make connections between these various representations and can build important understandings of mathematics and its applications.

S A M P L E R

BRIDGES
TO CLASSROOM
MATHEMATICS



A Mathematical/Pedagogical Unit

Measurement

Unit Overview

Activity	Description	Time	Materials
Measuring Length	Part 1: Participants establish personal reference measures for length and use these personal references to measure objects. Part 2: Participants measure using standard tools. The approximate nature of all measurements is discussed.	40 minutes	Assorted tools for measuring length (yard sticks, rulers, tape measures) Handout 1 Overheads 1 & 2 Chart Paper Blank Transparency
Area and Perimeter	Participants compare shapes with the same area and different perimeters. They also compare shapes with the same perimeter and different areas.	35 minutes	Square inch tiles Scissors Tape Handouts 2 & 3 Overheads 3 & 4
Weighty Investigations	Participants familiarize themselves with weights of common objects. They play a game to guess the weights of several objects. And, they estimate the weight of a million dollars.	45 minutes	Tools for measuring weight (spring scales, bathroom scales) At least 20 objects of varying weights Rubber bands, shopping bags, sandwich bags Large index cards Handouts 4 & 5

Summary of Session:

This session consists of three activities. First, participants use personal references and standard tools to measure length. They discuss the inevitability of error in all measurement. Next, they explore the relationship between area and perimeter, first by looking at shapes with fixed area and then by looking at rectangles with constant perimeter. The session ends with participants familiarizing themselves with weights of common objects by weighing, playing a game, and by estimating the weight of a million dollars.

Goals:

- To establish personal reference measures for length.
- To understand that all measurements are approximations.
- To investigate the relationship between area and perimeter of figures.
- To understand that a square is the quadrilateral that minimizes perimeter and maximizes area.
- To become familiar with the weights of common objects.
- To practice estimating weight.

Planning Ahead for the Session:

Read the Materials. Read the staff developer's guide for this session and the essay on measurement in the Mathematics Handbook.

Gather and Prepare Materials and Manipulatives. The activities in this session require a bit more advance planning than usual.

Spring scales and bathroom scales are needed for Activity 3. Bathroom scales can be borrowed for the seminar and you may have spring scales in science kits. Inexpensive and easy to use spring scales are available from ETA, 620 Lakeview Parkway, Vernon Hills, IL.

For Activity 3, you also need to gather at least 20 common objects of varying weights. Choose objects with weights that can be measured with the tools you have (spring scales and bathroom scales).

Prepare large index cards marked with the letters A through J on one side. On the reverse side of each card write a weight, e.g., 1/2 ounce, 1 ounce, 1 pound, 2.5 pounds, 5 pounds, 8 pounds, 10 pounds, 15 pounds, 20 pounds, 25 pounds. Randomize the weights on the cards so that the alphabetical order is not the order of weight. In this part of the activity, participants will gather objects and try to match the weights listed on each card, so make sure there are things in the workshop area that are somewhat heavy, e.g., large books.

Copy the Handouts. Since participants will be working in groups, one copy of each handout for every two participants should be enough.

Activity 1 — Measuring Length

Activity Synopsis:
Participants establish personal rulers for length, then use these personal rulers to measure the length of several objects. They then measure using standard tools. An examination of different groups' measurements leads to a discussion of the fact that all measurements are approximate.

Materials:
Length-measuring tools (rulers, yard sticks, tape measures)
Handout 1, Measuring Using Personal Rulers
Overhead 1, Measuring Using Personal Rulers
Overhead 2, Measurement Error
blank transparency
chart paper

Time:
40 minutes

Introducing the Session:
The first two activities in this session explore the familiar measurements of length and area. The third activity looks at the less familiar measurement of weight. You might begin with a few words about the session as a whole:

> **Today's session is about measurement. We'll begin with a length activity that involves several ideas that apply to all measurements. Then we'll investigate the area and perimeter of rectangles made with square inch tiles. Finally, we'll work a bit with measures of weight.**

Conducting the Activity:
This first activity might be a little surprising, at least to some participants. This has to do with an essential difference between counting and measuring: exact counts are possible, at least in theory, but all measurements are approximations. This activity is meant to help participants realize that measurement error is unavoidable.

Part 1: Measuring using Personal Rulers
You may want to start with a brief reminder that some of our familiar standard measures such as *foot* originated as measures of body parts, or, in the case of *mile*, as multiples of body measures. Refer to the Measurement essay in the Mathematics Handbook for more information.

What kinds of situations have you been in where you needed to measure something, but you didn't have a ruler or a tape measure?

How did you make the measurement?

Has anyone ever used a "personal ruler," for example, a body part or other object you always have with you?

Participant suggestions may include using a handspan to measure shorter lengths, or pacing out a length on the floor to measure a room length.

What personal ruler would you use to measure the width of this room, the length of the table, or a pencil?

Let participants discuss which body parts (or other things, e.g., a dollar bill) make the most sense for each of the above lengths. They should conclude that for floor or ground measurements, foot or pace length is sensible, but for other things, hand span, thumb width, width of two fingers, etc. are most useful.

Distribute Handout 1, Measuring Using Personal Rulers.

Now, each of you should find appropriate personal rulers to measure the three lengths on Handout 1. First, measure your personal rulers, then use those to measure the lengths on the list. Record your results.

Here's a tip: look for personal rulers that have "easy" measures to find multiples of, such as things that are close to an inch or 1/2 inch, foot or 1/2 foot, etc. For example, one of your fingers may be close to 4 inches long.

Display Overhead 1, Measuring Using Personal Rulers

<u>It would be best if participants all measured the same objects, but this may not be feasible with the pencil. They should, however, all measure the same room and table lengths.</u>

Demonstrate how to use the handout by measuring an object not on the list, e.g. the projector screen width, using your hand span as a personal ruler.

Object	Personal Ruler	Personal Ruler Length	Object Length using Personal Ruler
screen width	hand span	8 inches	60 inches

Make sure participants understand that the last column measurement is in standard units.

As participants complete Handout 1, circulate and note some of the participants' techniques and questions for the summary discussion.

Comparing Personal Ruler Measurements

When all are finished with the task, ask participants to consider the variation in measures.

Compare the three measurements you found with others at your table. Briefly discuss some of the reasons that these measurements vary.

Participants will not be surprised that the measurements made using personal rulers vary and will have numerous reasons for this, e.g., the length of a personal ruler was rounded to an “easy” number, they weren’t careful in moving their “rulers” along the objects, etc.

Since no unit of measure was specified for these measurements, it’s likely that the sizes of the units chosen by participants will vary. For instance, one person may have reported the length of the table to the nearest inch, whereas another may report it to the nearest 1/2 inch, or another to the nearest foot. If this is the case, have a brief discussion about reasonable units.

After a few minutes, draw the group together and wrap up this part of the activity.

In situations like this where we’re getting rough approximations, how do we decide on an appropriate unit of measure?

The following ideas should come up during this discussion. The choice of units depends on the

- purpose for the measurement,
- on the available tools, and
- on the situation.

This suggests that the choice of measurement units is in the hands of the measurer, i.e., there are no rigid rules. In general, however, it is better to be conservative and not to suggest a greater degree of precision than is reasonable. For example, in the case of the table measured with personal rulers, it is not sensible to report its length to the nearest 1/2 inch.

Part 2: Measuring using Standard Instruments

Post a sheet of chart paper or blank transparency labeled "Room Length".

We've just measured things using a variety of personal measuring tools and we found that we didn't all get the same results. Given our instruments, this is understandable.

Now let's see what happens if we use a standard instrument. Work with a partner and use whatever standard tool you wish to measure the length of this room. Measure to the nearest inch and report your results in inches on the chart paper.

Direct participants to the rulers, yard sticks, and measuring tapes.

As the pairs finish, ask them to write their results on chart paper (or on a blank transparency). When all have been collected, reorder them from highest to lowest in a single column.

Room Length (inches)
292
290
289
289
288
288
288
(and so on)

You can see that even using standard tools, we have some differences in our measurements.

Participants may be surprised that the measurements made using standard measuring tools vary so much.

Looking at this set of data, how might we decide on one good measure of this room's length in inches?

Some may suggest taking the value that occurs most often. Others may think about finding the mean. For this situation, it is probably best to find the middle number, the median. If this is unfamiliar to some, explain that the median is the middle number if the number of data pieces is odd. If it's even, the median is the average of the two middle numbers.

It is also true that even though we measured to the nearest inch, we could, in theory, have measured to the nearest 100th, 1000th, or 1,000,000th of an inch. In other words, we would never get it right. There is no right. There is only closer and closer.

Wrap up this part of the session by summarizing the key points, listed on Overhead 2.

Display Overhead 2, Measurement Error

Activity 2 — Area and Perimeter

Activity Synopsis:
Participants compare shapes with the same area and different perimeters. They also compare shapes with the same perimeter and different areas.

Materials:
- Square inch tiles (up to several dozen per participant)
- Handout 2, Grid (3/4 inch)
- Handout 3, Rectangles with a Perimeter of 20 Inches
- Overhead 3, Rectangles with a Perimeter of 20 Inches
- Overhead 4, Area and Perimeter
- Scissors
- Tape
- Chart paper

Time:
30 minutes

Conducting the Activity:
In the first part of this activity, participants explore the perimeter of different shapes made with square inch tiles. In the second part of the activity, participants investigate what different rectangles can be made with a given perimeter, again assuming the rectangles are made by putting square inch tiles together edge to edge.

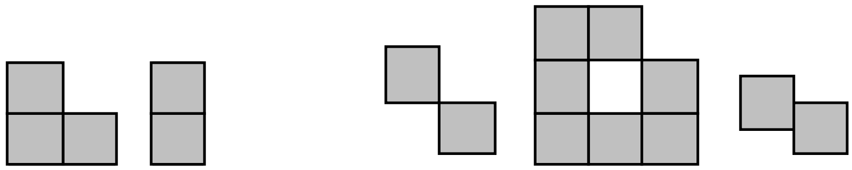
Shapes with Fixed Area

In this activity, we are going to look at a particular measure of length: perimeter. We will look specifically at the perimeters of different shapes having the same area.

You may need to take a minute to remind participants of the definitions of perimeter and area. Some participants may believe that if two shapes have the same area, then they must also have the same perimeter. This activity will help them to understand that shapes with the same area can have very different perimeters.

Distribute square inch tiles, Handout 2, Grid (3/4 inch), and scissors.

Ask the participants to work in pairs and to use the tiles to make several different shapes with an area of 12 square inches. Tell the participants to be sure to have each square tile touch another square tile on a whole side (see **Figure 1**). This will make it easier to calculate the perimeter of each shape. Shapes should not have holes.



These are OK.　　　　These are not OK.

Figure 1. Using square tiles.

Allow a few minutes for each pair to make several shapes with an area of 12 square inches.

Find the perimeter of each of the shapes you have made.

Then combine with another pair of participants to form a group of four. Compare all of your shapes.
Draw on grid paper the two shapes with the smallest perimeter and your two shapes with the largest perimeter. For sprawling shapes, you may need to tape two pieces of grid paper together. Then cut out the four shapes.

Ask the participants to tape these four shapes to a chart on the board. This chart should have two columns: one labeled “Smallest Perimeters” and one labeled “Largest Perimeters.” After all figures have been placed on the chart, conduct a discussion about the similarities and differences between the two groups. A completed chart might look like the one shown in **Figure 2**.

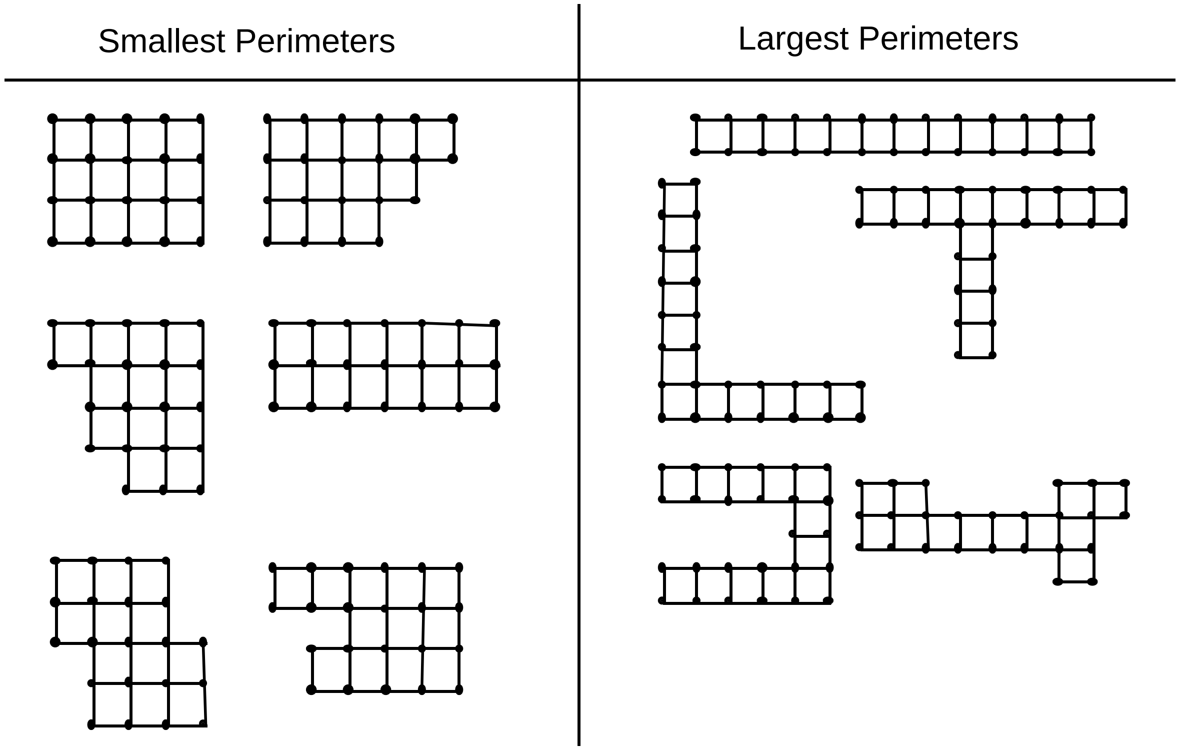


Figure 2. Smallest and largest perimeters with area 12 square inches.

Ask questions such as:

What do all the figures with the smallest perimeters have in common?

What do all the figures with the largest perimeters have in common?

What rectangles have the smallest perimeter? Why?

What rectangles have the largest perimeters? Why?

Given your experience with 12 tiles, what shape of area 16 square inches would have the smallest perimeter?

What 16 square inch shape would have the largest perimeter?

Use tiles to test your conjectures.

Check to see that all participants have determined that the 4 x 4 square (perimeter of 16) and the 1 x 16 rectangle (perimeter of 34) have the smallest and largest perimeters for a 16 square inch area.

Up to now, we have been looking at shapes that all have the same area. Now we are going to investigate rectangles that have the same perimeter.

In this task, you are going to find all possible rectangles with a perimeter of 20, assuming you are making them from square inch tiles put together edge to edge.

Give each participant a copy of Handout 3, Rectangles With a Perimeter of 20 Inches.

You may want to help the group get started. For example, tell them that one possible rectangle has a length of 9 and a width of 1. Therefore, the perimeter is 20 (9 + 9 + 1 + 1) and the area is 9 (9 x 1).

Many participants will be able to generate a list of all possible rectangles without drawing or using tiles. If they need to make the rectangles, you may want to suggest that they use dot paper (using the actual tiles can be awkward because different rectangles may use different numbers of tiles).

Use the chart on the handout to make a list of all possible rectangles with a perimeter of 20.

Note: In this activity, the rectangles should have integer side lengths. In other words, no rectangles should be made that have fractional side lengths.

Allow a few minutes for participants to complete the table.

Display Overhead 3, Rectangles with a Perimeter of 20 Inches.

Ask the participants to provide you with their findings. Use Overhead 3 to fill in all the possible rectangles with a perimeter of 20. A completed table will look like the following. You may want to point out the value in using a table to record all possible areas. It organizes your findings so that you can be sure that you have found all possibilities.

Rectangle Dimensions	Area of each Rectangle
1 x 9	9
2 x 8	16
3 x 7	21
4 x 6	24
5 x 5	25

Which rectangle has the largest area? What else can you interpret from this table?

Participants may note that the "skinniest" rectangle (the rectangle with the shortest width) has the smallest area. If they don't notice it, help them see that as the width and length get closer together in value, the areas of the rectangles increase. In other words, as the rectangles start to look more like a square, their areas increase.

Let's discuss how this relates to the part of the activity involving figures with a fixed area and different perimeters. What generalizations can you make about area and perimeter of rectangles?

Display Overhead 4, Area and Perimeter.

Discuss the following generalizations on Overhead 4:

- Of all the rectangles that can be made having the same perimeter, the one with the largest area is a square.
- Of all the rectangles that can be made having the same area, the one with the smallest perimeter is a square.

Activity 3 — Weighty Investigations

Activity Synopsis:

Participants familiarize themselves with weights of various common objects using different kinds of scales. They follow this with a game that involves estimating weight. Finally, they estimate the weight of a million dollars.

Materials:

- (3) Spring scales weighing up to a pound
- (1) Spring scale weighing up to 6 pounds
- (1) Spring scale weighing up to 11 pounds
- (2-3) Bathroom scales
- At least 20 common objects of varying weights, from about 1/2 ounce to about 20 pounds (e.g. keys, pencils, large and small books, calculator, bottle, waste basket, coffee mug, chair)
- Label 10 large index cards, one each, with the following weights: 1/2 ounce, 1 ounce, 1 pound, 2.5 pounds, 8 pounds, 10 pounds, 15 pounds, 20 pounds, and 25 pounds. Label each card on the blank side with the letters A-J
- Sandwich bags, plastic shopping bags, heavy-duty rubber bands for holding items to be weighed
- 4-5 calculators
- Handout 4: Estimating and Measuring Weight
- Handout 5: The Weight is Right Game Estimates
- 3 or 4 (in case of ties) Pound bags/boxes of cookies or candy used as prizes for the game

Time:

45 minutes

Introduction to the Activity:

In this activity, participants investigate objects of varying weights in order to gain an intuitive understanding of the approximate weights of objects.

Note: This activity ignores the distinction between mass and weight and uses the terms interchangeably. (Refer to the Bridges Mathematics Handbook for an explanation of the difference between mass and weight.)

Conducting the Activity:

In Activity 1, we established personal reference measures for measuring length. However, we do not carry around personal reference measures for measuring weight. Therefore, it is much more difficult to estimate the weights of objects. In fact, many people have trouble determining the weight of an object because they have not familiarized themselves with what particular weights feel like. For example, if you pick up a dictionary, can you guess how much it weighs?

The goal of this activity is to familiarize yourself with varying weights so that you develop your intuition about how much objects weigh.

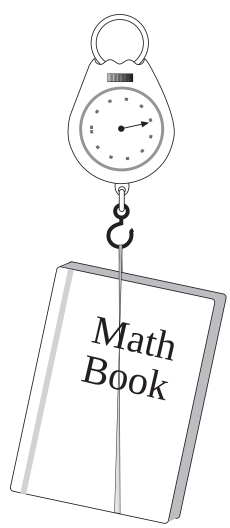
Have the participants work in groups of three for this activity.

Distribute Handout 4, Estimating and Measuring Weight

To begin, each group should choose 5 of the objects displayed. First estimate the weight of, then weigh the object. Choose a variety of light and heavy things and record your results on the handout.

Show the locations of the spring and bathroom scales and the objects to be weighed. If you have scales of different capacities, tell the participants the capacity of each type.

As groups start to work, you may need to demonstrate the use of rubber bands or sandwich bags to attach objects to the hooks on the spring scales.



While circulating, suggest that participants weigh some objects (at least 5 pounds) on both spring and bathroom scales. They may see a difference in their results.

It is likely that the bathroom scales are especially inaccurate at small weights. To test whether bathroom scales are more accurate at heavier weights, e.g., over 100 pounds, ask a volunteer to weigh him/herself with and without the object in question and subtract to find the weight of the object. This is also an easy way to weigh things that are cumbersome on a bathroom scale.

When all groups have completed their weighing, briefly go over their findings.

Did your estimates of weight improve with experience?
Compare your results on this list with your list of length measures.
Which are you better at estimating, length or weight?
Why do you think this is the case?

Compare a few of the actual weight measures different groups got for the same object. If there are differences, ask the participants to suggest reasons for this. This is a good time to talk about the inaccuracies of many scales. It is likely, for example, that different scales in the workshop will give differing results. Do they believe that scales are less accurate than rulers or measuring tapes?

"The Weight is Right" Game

Participants will work in groups of three again, but you may want to switch group members.

Give each group an index card with a weight on one side and a letter designation on the other. Make sure only the letter side is showing.

We're going to play a game called "The Weight is Right." Here are the rules.

Each group has a card with a letter on it. Without showing it to other groups, look at the opposite side of the card. You will see a weight. Your task is to find an object or a set of objects that equal that weight.

You may not use any of the objects from the first activity. You may use any of the scales to weigh your collection.

When you've reached your goal, place your collection together with your card, letter side up, on a table. When all groups have done this, I'll give you the rest of the rules.

Distribute Handout 5, The Weight is Right Game Estimates.
Have available some large, plastic shopping bags to hold collections for weighing.

Now, each group should visit each station. You're to estimate the total weight of the set of objects. You can use anything you want that might help you make a good guess, EXCEPT SCALES.

Record your estimates on Handout 5.

After the lists are completed, draw the groups together to determine the winning team. Ask a volunteer to turn over the cards. As each card is revealed, the team with the best estimate gets a point. The team with the most points wins the game.

To wrap up this activity, briefly discuss the following:

When we worked with length, we developed reference measures such as hand span to help us estimate. What are some things we can do to improve our weight estimations?

Length is something we can see and therefore we can visually gauge the approximate length of an object. Furthermore, using body parts or other personal references facilitates our sense of length. On the other hand, we do not carry around personal reference measures for weight and, therefore, the one way to determine them without a scale is to experience weighing a lot of objects. Participants may have some ideas based on their experiences. For example, experienced cooks may have good food estimation skills based on the size/volume of meat, produce, etc.

A Million Dollars Weighs?

To conclude this session, ask teachers to work in groups and ponder the following:

You've all seen in movies or books the following scenario: a kidnapper or robber says, "Fill a briefcase with a million dollars in small, unmarked bills."

About how much would a million dollars weigh? Approximate the weights for a million dollars in $1 bills and in $20 bills. Use anything you can find to help you, including all scales.

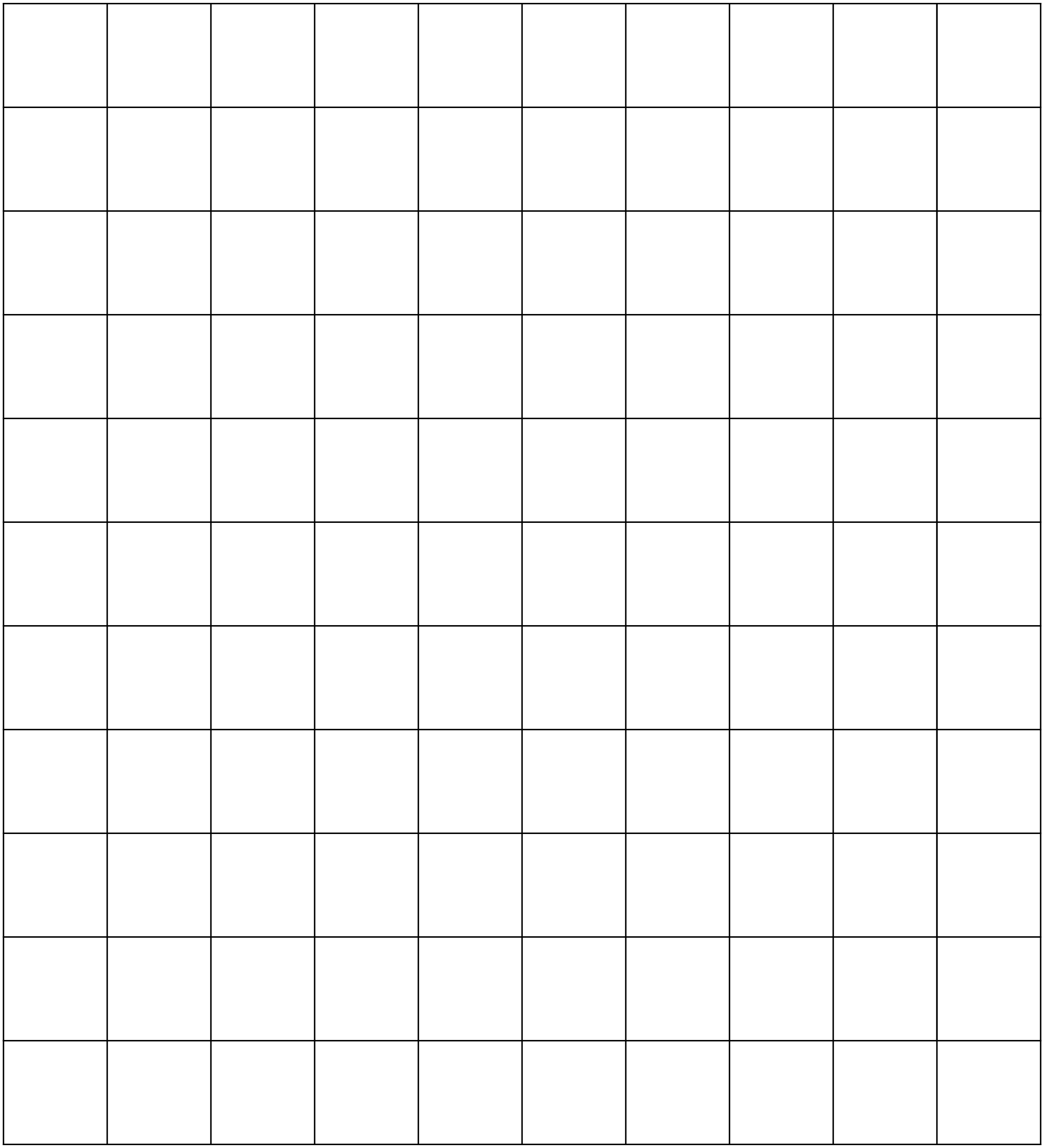
Participants will probably figure out rather quickly that 20 bills weigh about an ounce. Thus, a million $1 bills would weigh about 3,000 pounds (3,125 is the exact calculation, but is probably too precise an estimate and should be rounded). A million dollars in $20's weighs about 150 pounds.

When participants have figured this out, talk a bit about how hard it would be to carry a million dollars and would it really fit in a briefcase?

Measuring Using Personal Rulers

Measurement	Personal Ruler	Personal Ruler Length	Object Length using Personal Ruler
screen width	hand span	8 inches	60 inches
table width			
room width			
pencil length			

Grid



Rectangles With a Perimeter of 20 Inches

Rectangle Dimensions	Area of each Rectangle

Estimating and Measuring Weight

Object	Estimated Weight	Measured Weight

"The Weight is Right" Game Estimates

Object(s)	Estimated Weight	Measured Weight	Points
A			
B			
C			
D			
E			
F			
G			
H			
I			
J			
		Total	

Measuring Using Personal Rulers

Measurement	Personal Ruler	Personal Ruler Length	Object Length using Personal Ruler
screen width	hand span	8 inches	60 inches
table width			
room width			
pencil length			

Measurement Error

All measurements are approximations. Measurement error is inevitable. It is possible to count things exactly but it is impossible to measure things exactly because:

- There is variation no matter what measurement technique is used. Variation in measurements arise from imperfections in measuring tools and techniques.
- Since you can always imagine a ruler with more divisions, you can always imagine getting a more precise measure.

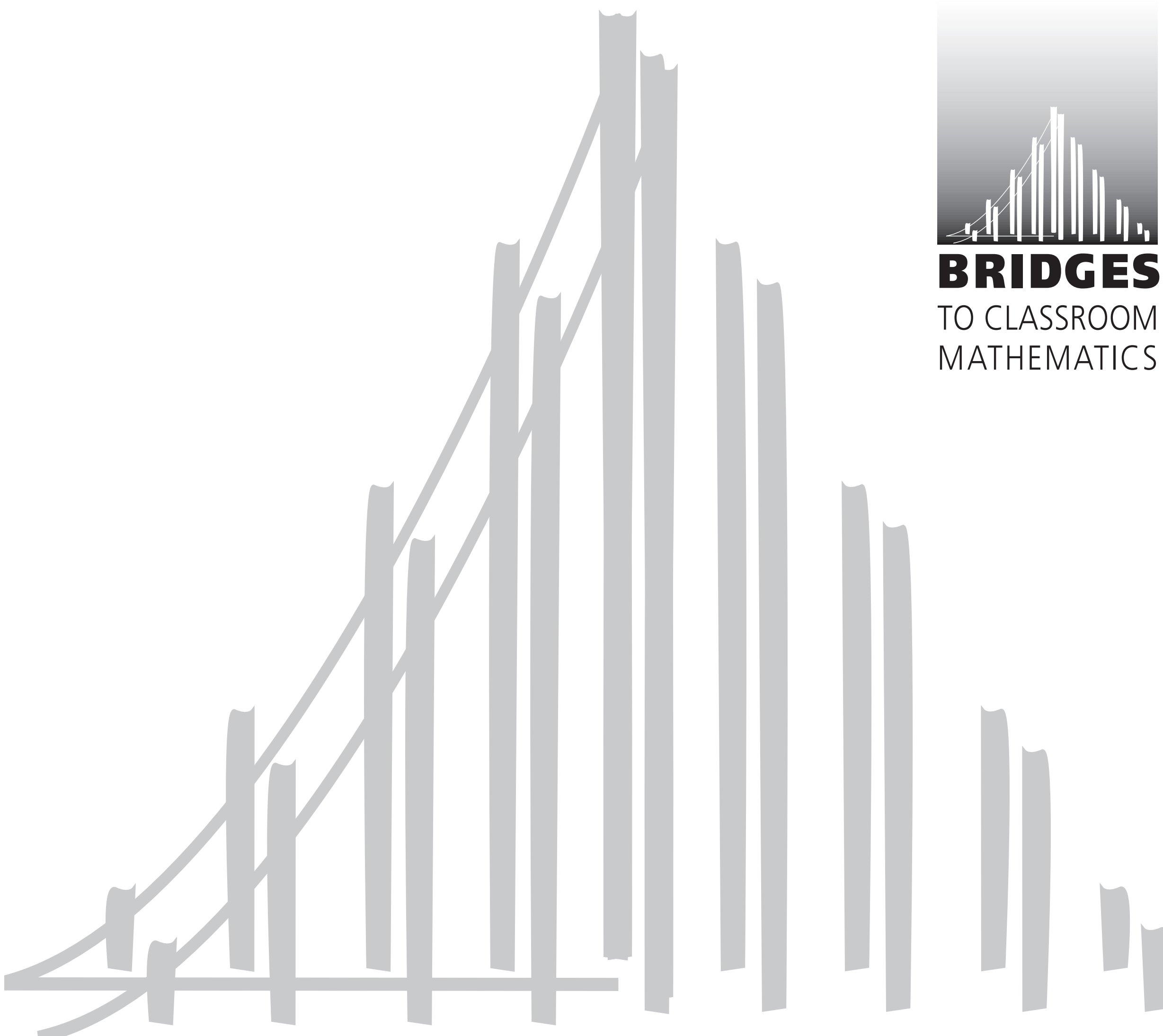
Rectangles With a Perimeter of 20 Inches

Rectangle Dimensions	Area of each Rectangle

Area and Perimeter

- Of all the rectangles that can be made having the same perimeter, the one with the largest area is a square.
- Of all the rectangles that can be made having the same area, the one with the smallest perimeter is a square.

S A M P L E R



BRIDGES
TO CLASSROOM MATHEMATICS

An *Everyday Mathematics* Unit

Procedures for Multiplying & Dividing in *Everyday Mathematics*

Unit Overview

Activity	Description	Time	Materials
Division Dash	Participants play a game that raises several questions about how to develop students' facility with multiplication and division.	25 minutes	Calculators with square root function (1 per participant) Overhead calculator with square root function Handout 1 Overhead 1
Stations	Participants rotate among four stations, three with low-stress algorithms for multiplication and division, and one with a game.	75 minutes	Handouts 2–9 Overhead 2 Station Task Cards Scratch paper Base ten pieces Red, blue, and green markers or crayons Calculators with a constant function Decks of number cards
Summary Discussion	Participants discuss session activities and the Everyday Mathematics approach to multiplication and division.	20 minutes	Handouts 2 and 10 Overheads 2–5

Summary of Session:

This session begins with a game in which players divide random two-digit numbers by random one-digit numbers. To carry out the divisions, players either devise their own methods or adopt a strategy from *Everyday Mathematics*. After the game, participants work at stations with activities from *Everyday Mathematics*. Several of these stations explore "low-stress" paper-and-pencil algorithms for multiplying and dividing multi-digit numbers. The session concludes with a discussion of the *Everyday Mathematics* approach to developing appropriate multiplication and division skills.

Goals:

- To help participants understand the balance among written, mental, and machine computation in *Everyday Mathematics*.
- To give participants experience with the *Everyday Mathematics* approach to multiplication and division of multi-digit numbers, especially the use in the curriculum of standard, invented, and alternative algorithms.
- To examine several low-stress multiplication and division algorithms from *Everyday Mathematics*.

Planning Ahead for the Session:

This session is closely related to Multiplication and Division Concepts in *Everyday Mathematics*. The two sessions together give a broad overview of the treatment of multiplication and division in *EM*, so participants should attend both. Neither session depends on the other, so you can do them in whatever order you choose. If you have no preference, then do "Concepts" first and "Procedures" second.

Read the Materials. Read the Staff Developer's Guide for both sessions(as mentioned above) before you conduct either. Portions of the "Operations" essay in the Bridges Mathematics Handbook also support these sessions. You might also read the *Everyday Mathematics* teacher materials for activities from the session. Division Dash and the mental division technique it uses are taken from Lesson 62 of *Fifth Grade Everyday Mathematics*. Getting to One is introduced in Lesson 39 of *Fourth Grade Everyday Mathematics*. Array Multiplication is an exploration from Unit 11 of *Third Grade Everyday Mathematics*. Multiplication Wrestling is introduced in Lesson 42 of *Fourth Grade Everyday Mathematics*. The Teacher's Reference Manual has several pertinent sections. Those on Multiplication and Division, Algorithm Development, Calculators, Mental Computation, and Games are particularly relevant.

Gather and Prepare Materials and Manipulatives. Two games in this session require more than simple four-function calculators. Division Dash requires calculators with a square root key and Getting to One requires calculators with a constant function. The only other special materials required are the base0-ten pieces and colored markers that are used for Array Multiplication at Station B and the number decks for Multiplication Wrestling at Station E.

Copy the Handouts. Most of the handouts for this session are used only at the stations and are not consumable, so you will not need copies for each participant. Depending on the size of your group, 3 to 5 copies of these should be enough. Other handouts used at the stations are to be written on, so you will either need to laminate several copies or make enough for each participant to have one. Finally, there are a few handouts that should be copied for every participant.

The numbers below are for a workshop with 25 participants.

Handout 1 (Division Dash)..15
Handout 2 (Questions about …) 25
Handout 3 (Lattice Multiplication) ....3 laminated or 25 plain
Handout 4 (Blank Lattices)............................3 laminated or 25 plain
Handout 5 (Array Grid) 3
Handout 6 (Array Record Sheet) ....3 laminated or 25 plain
Handout 7 (Rules for Getting to One Game) 3
Handout 8 (Multiplication Wrestling)...3

Handout 9 (Multiplication Wrestling, cont.)....................................3
Handout 10 (Alternative Algorithms)..25

Note on Organizing the Session
As an alternative to the stations in this session, you can organize the activities instead as a series of whole-group exercises. This may be preferable if there is only one leader for the session, especially if the participants are relatively inexperienced. In this case, the staff developer can provide more direction and can discuss each activity in turn. If you choose this option, however, you will need more materials and more copies of the handouts. Also, the whole-group approach is more demanding on the staff developer and limits the participants' autonomy.

If you decide to do the stations as whole-group activities, here is a possible order:

Division Dash
Lattice Multiplication
Array Multiplication
Partial Product Multiplication
Multiplication Wrestling
Low Stress Division
Getting to One

Since the staff developer will be leading the activities, the participants will not need some of the station task cards; other cards will need to be copied. The questions on Handout 2 need not be distributed to the participants either; they can be raised by the staff developer during the discussion of each activity.

Copy the handouts as follows:

HO 1 (Division Dash)...15
HO 2 (Questions about …)..25 or 0
HO 3 (Lattice Multiplication)...25
HO 4 (Blank Lattices)..25
HO 5 (Array Grid)..25
HO 6 (Array Record Sheet)...25
HO 7 (Rules for Getting to One Game)...15
HO 8 (Multiplication Wrestling)...15
HO 9 (Multiplication Wrestling, cont.)...15
HO 10 (Alternative Algorithms)...25

Station Card A/2 (Egyptian Multiplication)...................................25
Station Card B/1 (Array Multiplication)..25
Station Card B/2 (Partial Product Multiplication)..........................25
Station Card C/1 (Low-Stress Division)...25

You will also need enough materials for all participants to do the activities at once. This is most problematic for the base 10 pieces for Array Multiplication.

Activity 1 — Division Dash

Activity Synopsis:
Participants play Division Dash, a game first introduced in *Fourth Grade Everyday Mathematics*. Besides being fun and showing how a game can provide skills practice while also promoting concept development, Division Dash is a good example of the *Everyday Mathematics* approach to computation.

Materials:
Handout 1 (1 for every 2 participants)
Overhead 1
Calculators with a square root function (1 per participant)
Overhead calculator, ideally the same model as participants'

Time:
25 minutes

Conducting the Activity:
You might want to begin with a few words about the session as a whole:

This session is about how *Everyday Mathematics* helps students develop a variety of mental and paper-and-pencil methods for multiplying and dividing. The session has two parts. First we'll play a game that involves lots of mental arithmetic. Then we'll have stations with selected activities from the curriculum.

A Mental Division Procedure
Before you introduce Division Dash, ask participants to solve a two-digit by one-digit division problem without using a standard algorithm:

Suppose you didn't know long division.

How could you solve a problem like 75 ÷ 6?

Take a minute or two and try to think of at least two ways.

As you discuss participants' approaches, you might ask which could reasonably be used by fourth- or fifth-grade students. You might also ask how a second-grade student could solve the problem. As participants present their methods, point out that some are purely mental while others use tools like paper and pencil or counters.

After participants have shared their methods for solving 75 ÷ 6, present the following technique, which is described more fully in Lesson 62 of *Fifth Grade Everyday Mathematics*.

Here's a method for solving problems like 75 ÷ 6 mentally. Break apart the larger number into two or more "friendly" numbers that are more easily divisible by the single-digit number. For example, you might break 75 into 60 and 15. Then, since 60 ÷ 6 is 10 and 15 ÷ 6 is 2 with remainder 3, we have 75 ÷ 6 equals 10 + 2 with a remainder of 3, or 12 with remainder 3.

How else could 75 be broken apart so that 75 ÷ 6 is easier to solve mentally?

After participants suggest other ways to break apart 75, ask them to try the technique on several other problems:

Try this method on 96 ÷ 3 and 71 ÷ 4.

Now make up some two-digit by one-digit division problems and solve them with this method.

Make sure participants are comfortable with this division technique before you begin the game Division Dash.

Division Dash

Now I want you to play a game using this mental division procedure. Since the rules are a little complicated, I'd like to play a round or two on the overhead.

Distribute calculators to all participants.

If you have an overhead calculator that's identical to participants' calculators, then ask them to follow along. **Overhead 1** shows the rules for the game.

Display Overhead 1.

Demonstrate how to play the game, especially how to use the calculator to generate a series of random division problems and how to keep score. With only one overhead calculator, you won't be able to demonstrate both sides of an actual game, but you should be able to clarify the play of one side, which should be enough.

After your demonstration, ask participants to pair up to play the game. **Handout 1** gives the complete rules and several examples. Emphasize that participants should try to solve the problems mentally, not by long division or with the calculators.

Distribute Handout 1.

Try to use the strategy we just discussed or some other mental method to do the divisions in this game. Do not use long division or a calculator. You can play to 50 instead of 100.

Let participants play for a few minutes. It is not necessary that every group finish a complete game. Playing to 50 should be enough to get the idea.

Discussion

Division Dash raises several issues worth considering.

Here are two ways you might stimulate some discussion.

In *Fifth Grade Everyday Mathematics*, Division Dash appears immediately before the introduction of a long division algorithm. This means students must use what they know to solve problems that would traditionally be solved by long division. What are the possible advantages and disadvantages of this approach?

or

At the beginning of this session I asked you to solve 75 ÷ 6 by a method other than long division. After you had a chance to devise your own methods, I showed you a method from *Everyday Mathematics*. Then you played a game in which you could use either your own method or the one I showed you.

Compare this mixture of "invented" and taught procedures with the more traditional approach where students are first shown a method and then practice applying that method. What are some advantages of each approach?

Leading this discussion and the summary discussion at the end of the session may be challenging. Some participants may feel strongly about teaching "standard" multiplication and division algorithms in a traditional way. In our experience, it is usually best not to argue with people who feel this way: a direct approach is unlikely to change any minds. Instead, you might point out that *Everyday Mathematics* encourages teachers to use their professional judgment:

> "There is often outside pressure for children to master specific standard algorithms. Indeed, this may be your own strong preference. In either case, we hope that you will do what is best suited to your own situation—our aim is to help teachers, not to impose our own ideas or demands on them." (*Third Grade Everyday Mathematics* Teacher's Manual and Lesson Guide, p. 169.)

In the course of the discussion, either now or at the end of the session, be sure to cover the following points:

- **Conceptual understanding.** An important reason for giving students problems that they cannot solve by the immediate application of a memorized procedure is to develop their conceptual understanding of the operations involved. Requiring students to think rather than simply to apply a memorized procedure certainly takes longer and is more difficult for everyone involved. But thinking problems through will help students develop better understandings and ultimately better skills too.

- **Common sense.** Perhaps the most important reason for encouraging students' invented methods is so that they come to perceive school mathematics as an extension of common sense, rather than as opposed to common sense. In traditional arithmetic programs, a growing separation between school mathematics and common sense leads to many nonsensical mistakes. Encouraging children's own methods can help prevent that separation.

- **Mental arithmetic.** Another reason to emphasize students' invented methods is to help them develop their mental arithmetic skills.

Do not let this discussion go on too long, since there will be another discussion at the end of the session.

Activity 2 — Stations

Activity Synopsis:
Participants rotate among four stations with activities from *Everyday Mathematics*. These station activities provide first-hand experience with the curriculum's approach to multiplication and division, especially where that approach may be unfamiliar to the workshop participants.

The four stations are (A) Ancient Algorithms, (B) Arrays & Partial Products, (C) Division, and (D) Multiplication Wrestling.

Materials:
Station A
- Station Task Cards for Activities 1 & 2
- Handouts 3 & 4
- Calculators (2)
- Scratch Paper

Station B
- Station Task Cards for Activities 1 & 2
- Handouts 5 & 6
- Base 10 Pieces
- Green, red, and blue colored markers, pencils, or crayons
- Scratch Paper

Station C
- Station Task Cards for Activities 1 & 2
- Handout 7
- Calculators (2)

Station D
- Station Task Cards for Activity 1
- Handouts 8 & 9
- Decks of Number Cards (2)
- Calculators (2)
- Scratch Paper

Time:
75 minutes

Conducting the Activity:
Ideally, the stations should follow naturally from the discussion of Division Dash. During that discussion, one issue may be what to do with students who fail to

develop effective multiplication and division procedures on their own. Everyday Mathematics provides a variety of "low-stress" algorithms that can serve as fall-back procedures for such students; several of the stations below involve such algorithms.

The next part of the session will be devoted to stations with selected student lessons from *Everyday Mathematics*. Several of these stations have "low-stress" multiplication and division algorithms from the curriculum.

Before participants begin, take a moment to prepare them for the discussion at the end of the session.

Distribute Handout 2.

As you work at the stations, discuss the questions on the station cards. Also think about the questions on Handout 2. We'll discuss these issues at the end of the session.

The Stations

The station task cards specify the activities for each station. The stations are designed to be self-explanatory, so that you should be free to observe participants and to make notes for the summary discussion.

Participants should work in small groups, beginning at any station and moving to another station roughly every 15 minutes. Ask participants to erase any laminated worksheets and to leave the station in order for the next group.

Note: The activities at the stations can, if you wish, be done sequentially with the whole group. See "Planning Ahead for the Session" above for suggestions about this option.

Activity 3 — Summary Discussion

Activity Synopsis:

Participants discuss the Everyday Mathematics approach to multiplication and division of whole numbers. The discussion should explore questions from the station task cards and Handout 2, but other issues may also be raised.

Materials:

- Handouts 2 & 10
- Overhead 2, Questions about Multiplying and Dividing in Everyday Mathematics (= Handout 2)
- Overhead 3, Lattice Multiplication
- Overhead 4, Place Value in Lattice Multiplication
- Overhead 5, One Cell's Value

Time:

20 minutes

Conducting the Activity:

Because the Everyday Mathematics approach to methods for multiplication and division differs considerably from what is traditional, it is important to close this session by providing an opportunity for participants to discuss the day's experiences in light of their current practice.

You may want to begin the discussion by reviewing some of the station activities about which teachers had questions. You may want to distribute Handout 10, which has examples of three alternative algorithms from the station activities. Of these, Lattice Multiplication is probably the most puzzling.

Lattice Multiplication

Lattice Multiplication is a very old method, dating back to the Middle Ages. Once the lattice has been drawn, it is also a very efficient method. And although Lattice Multiplication is easy to learn, it is not easy to understand.

The reason Lattice Multiplication works is illustrated in **Overheads 3–5**. The geometric arrangement of the numbers automatically takes care of place value, so that the user need know only the single-digit multiplication facts.

Discussing Procedures for Multiplying and Dividing in *Everyday Mathematics*

Next you might ask for other questions or comments about the activities in the session. Let the discussion be free-flowing, and try to keep the participants talking to each other as much as possible. Also try to focus the discussion on issues related to this session. Let participants know that the *Everyday Mathematics* "scope and sequence" for multiplication and division is addressed in another session.

You can use the questions on Handout 2/Overhead 2 to redirect the discussion if it begins to wander. Station Cards A-1, A-2, C-1, C-2, and D-1 also have the questions that are worth discussing.

It might be fun to ask participants to prepare a role-play or debate about the questions on Handout 2. For example, participants could role-play an open-house discussion in which a parent requests an explanation for the use of alternative algorithms. Or participants could debate the merits of using games for practice. Or participant pretending to be a representative of the Everyday Learning Corporation could debate another pretending to be from the publisher of a more traditional program. Each "sales rep" could claim a better approach to computation.

If participants do not raise the following points during the discussion, then you should.

- *Everyday Mathematics* takes a moderate approach to developing multiplication and division skills. The program attempts to balance mental, paper-and-pencil, and calculator methods of computation.

- The program recognizes that some students do not develop effective methods on their own, especially for multiplication and division. The low-stress alternative algorithms in the curriculum provide these students fall back methods they can adopt.

- The low-stress algorithms in *Everyday Mathematics* are easier to carry out because there is less to remember and are also sometimes easier to understand. Partial Product Multiplication, for example, is much easier to understand than the traditional long multiplication algorithm, although it is less efficient. But if efficiency is the goal, any paper-and-pencil method is inferior to a calculator.

In its approach to developing computational facility, *Everyday Mathematics* differs significantly from traditional programs, so that many teachers and parents may be initially skeptical. In this case, as mentioned above, be sure to let participants know that the *Everyday Mathematics* authors respect their judgment and encourage them to adjust the program to fit local conditions. Your professional judgment and experience will likewise be essential as you lead this workshop; these are topics on which many people have strong opinions that will not be easily changed.

Division Dash

Materials: A calculator for each player

Number of Players: 1 or 2

Object of the Game: To reach 100 or more

Directions:

1. Each player chooses a number at least three digits long and enters it into the calculator.

2. Each player presses the square-root key [√‾]. If the number in the display has only one or two digits, start over. If the final digit is 0, start over. Each player uses the final digit as the **divisor** and the two digits before the final digit as the **dividend**.

3. Divide the dividend by the divisor. Record the whole-number part, or **quotient**. Ignore the leftover part, the **remainder**. Players calculate mentally or on paper, not on the calculator.

4. Players take turns repeating Steps 2 and 3, starting with whatever number is in the display. Each player keeps track of the sum of his or her quotients. The first player to reach a sum of 100 or more wins.

Example: Player enters 5678.

	Quotients
Press √‾ and get 75.3 5 2 5 0 4 9 4 .	
Divide 49 by 4. Record the result. (12, ignoring the remainder)	12
Press √‾ again and get 8.6 8 0 5 8 2 0 6 2 .	
Divide 6 by 2. Record the result. (3)	3
Press √‾ again and get 2.9 4 6 2 8 2 7 5 3 .	
Divide 75 by 3. Record the result. (25)	25
	40

If there is only one player, the object of the game is to reach 100 or more by solving the fewest number of division problems.

Challenge Version:

Attach 0 to each divisor and dividend. For example, if the display is 2.946182827, the problem would be 820 divided by 70.

Questions about Multiplying and Dividing in *Everyday Mathematics*

1. How would you explain to parents the *Everyday Mathematics* use of invented and alternative algorithms instead of traditional long multiplication and long division?

2. What do you think about the balance of mental arithmetic, paper-and-pencil skills, and calculators in *Everyday Mathematics*? How well are these balanced in the traditional curriculum?

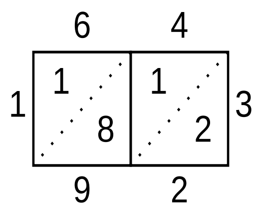
3. What do you think about using games to practice computational skills?

Lattice Multiplication

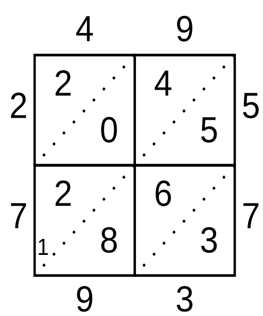
Lattice Multiplication is an old method. Figure out how it works by studying the problems in Column A. Then try to use lattice multiplication to solve the problems in Column B.

Column A

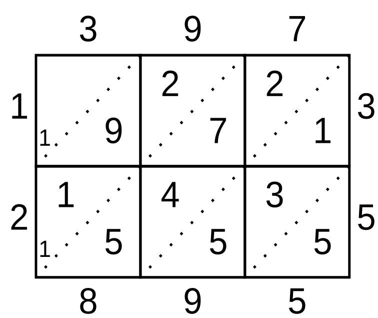
1. $3 \times 64 = 192$



2. $57 \times 49 = 2793$

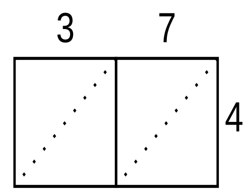


3. $35 \times 397 = 12{,}895$

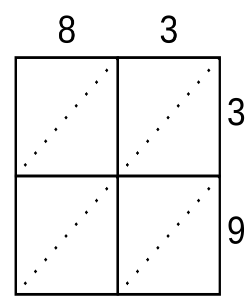


Column B

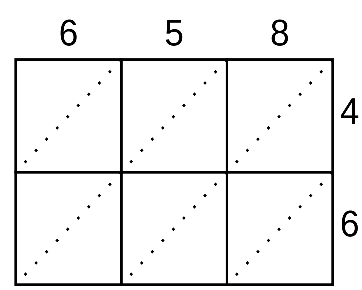
1. 4×37 = __________



2. 39×83 = __________



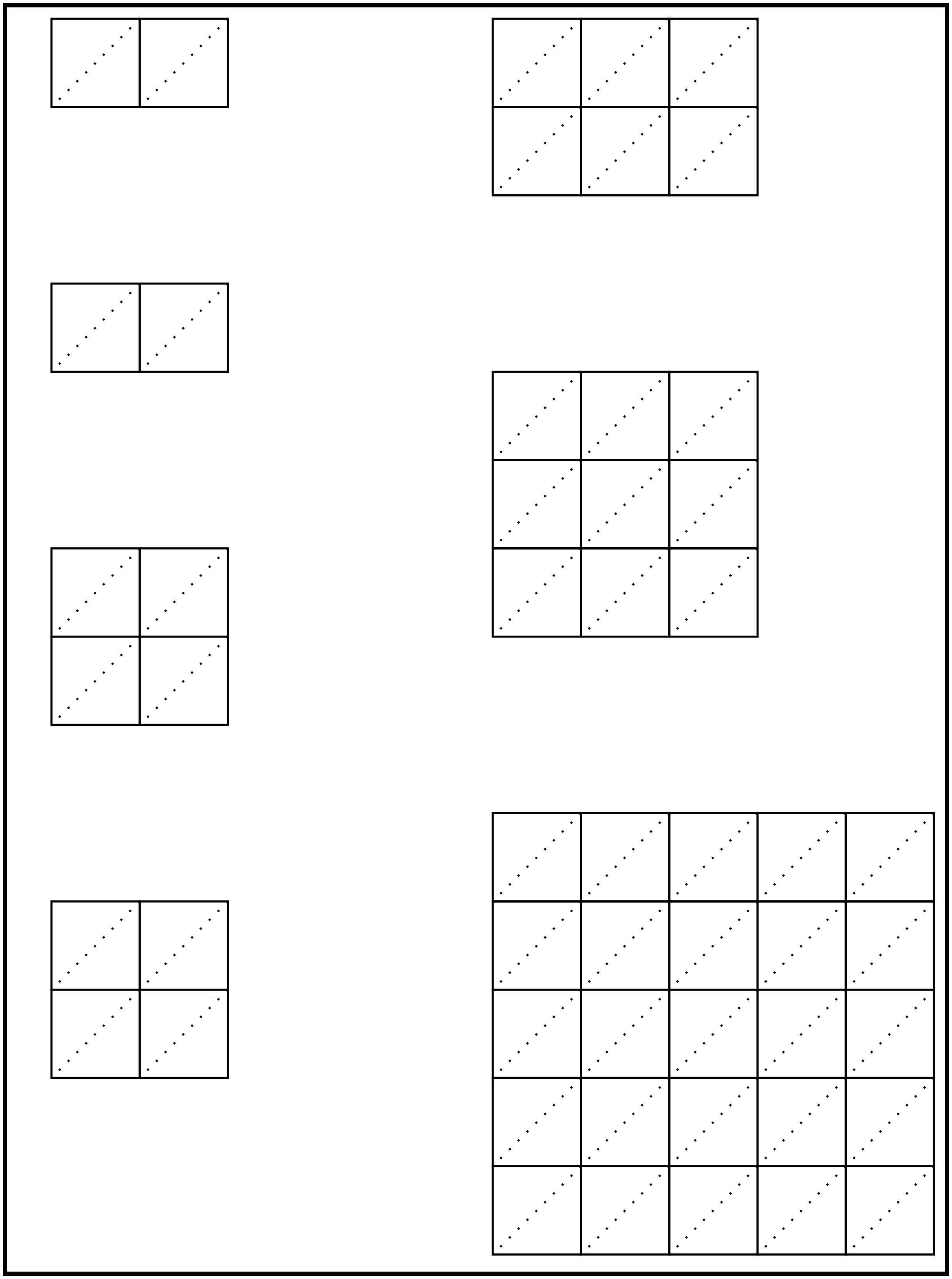
3. 46×658 = __________



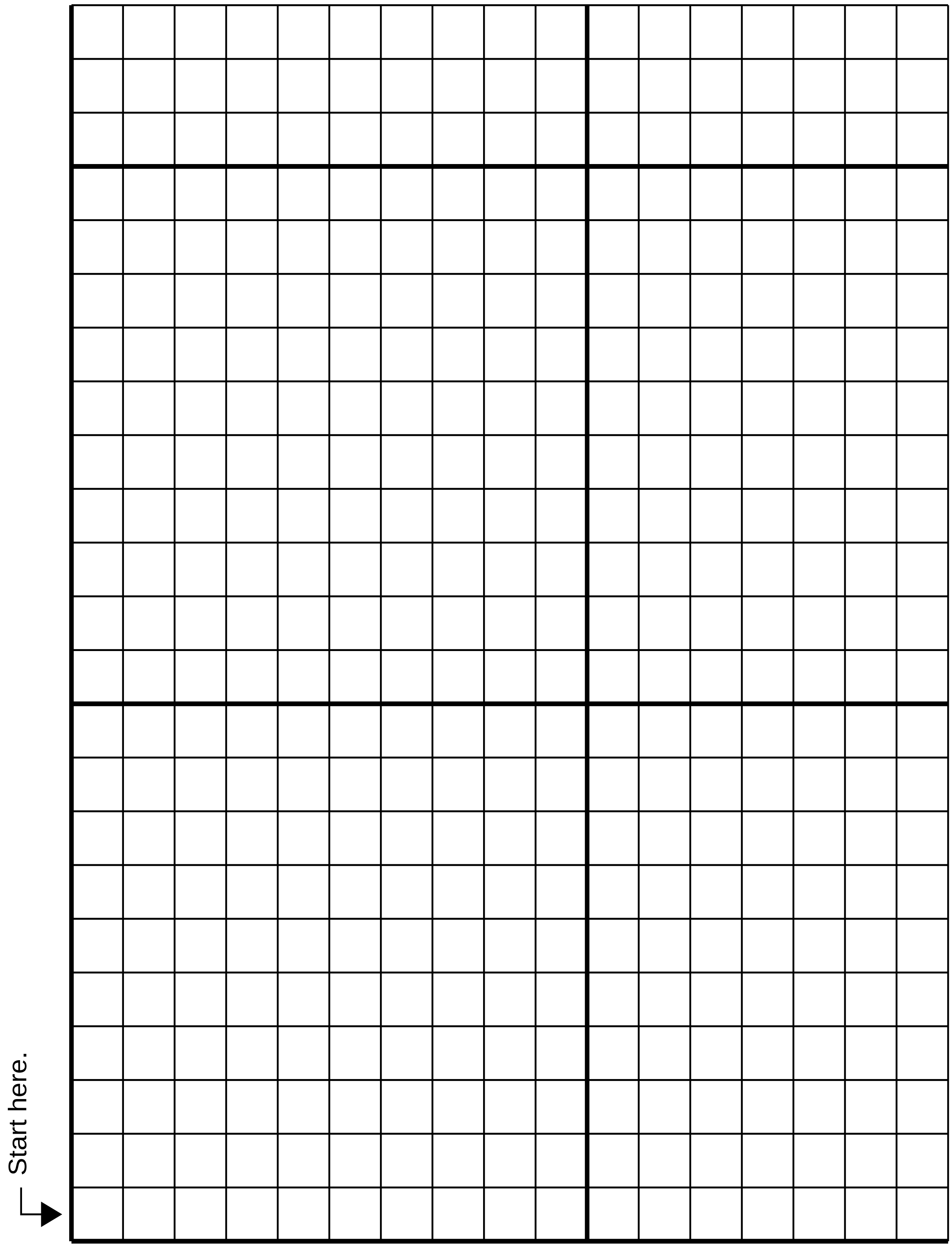
A note from history about lattice multiplication
The search for ways to record computation started in India, perhaps about the eleventh century. This method of multiplication was known to the Arabians and probably was copied from the Hindus. Fifteenth-century writers in western Europe included it in their printed books.

The first printed arithmetic book appeared in Treviso, Italy, in 1478. Luca Pacioli of Italy listed eight different ways to do multiplication in his book called the *Suma*. He called this way "lattice multiplication." The name suggests the gratings placed in Venetian windows to keep people from looking through them.

Blank Lattices

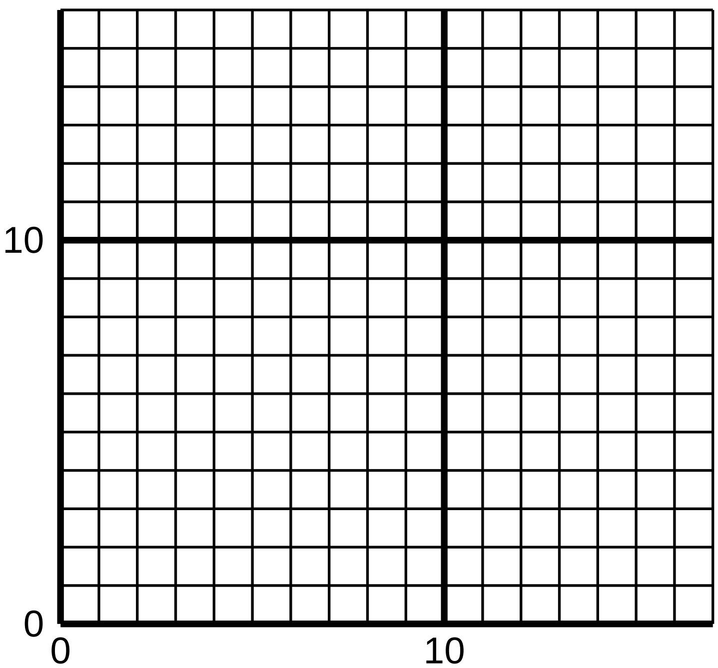


Array Grid



Array Record Sheet

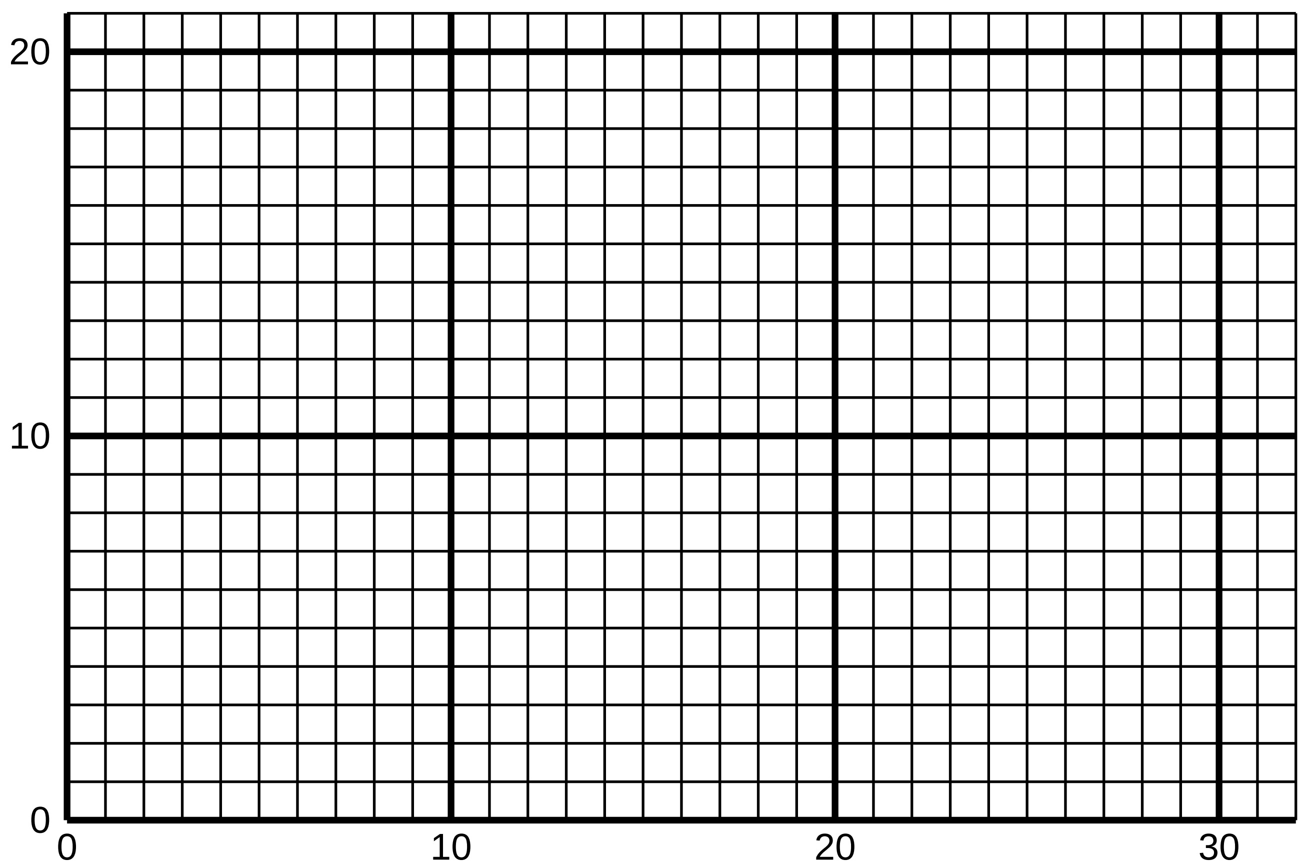
1. How many squares are in a 14-by-12 array?



Total squares = __________

14 × 12 = __________

2. How many squares are in a 21-by-24 array?



Total squares = __________

21 × 24 = __________

Rules for *Getting to One* Game

Materials: calculator

Number of players: 2

Object of the game:
One player chooses a mystery number. The other player tries to guess the number in as few tries as possible. Players then trade roles. The player who guessed the mystery number in fewer tries wins the round.

Directions:
1. Player A chooses a mystery number less than 100.
2. Player A then secretly enters the number in the calculator and divides it by itself. For example, if the mystery number is 65, Player A enters 65[÷] 65 [=]. (on calculators with a [K] key, enter 65 [÷] 65 [K] [=].) The result should be 1.
3. Player B guesses the mystery number and, without clearing the calculator, enters the guess and [=] in the calculator.
   - If the calculator shows a number less than 1, then the guess was too low.
   - If it shows a number greater than 1, then the guess was too large.
   - If it shows a 1, then Player B guessed the mystery number.

Player B enters guesses until the result is 1. Player A keeps track of the number of guesses. **Do not clear the calculator until the number has been guessed.**

Example: Mystery number = 65

Player B enters:	**Calculator shows:**	
55 [=]	0.8461538	too small
70 [=]	1.076923	too big
67 [=]	1.0307692	too big, but closer
65 [=]	1	Just right!

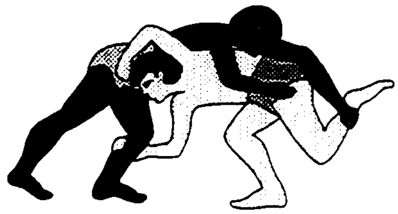
It took Player B four tries to guess the mystery number.

Scoring:
One way is to play 5 rounds in which there were no ties. The player who won more rounds wins the game.

Another way is to play 5 rounds and to keep track of the number of guessesfor each round. The player with fewer guesses in all wins the game.

For a harder version of the game, allow mystery numbers up to 1000.

Multiplication Wrestling



Materials: a deck of 0–9 number cards (4 of each number for a total of 40 cards)

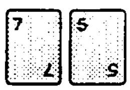
Number of players: 2

Object of the game: To get the largest product of two 2-digit numbers.

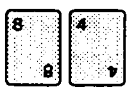
Directions: Shuffle the deck of cards and place it facedown. Each player draws 4 cards and forms two 2-digit numbers. There are many possible combinations of 2-digit numbers. Each player must pick a pair of numbers to use.

Example:

Player 1

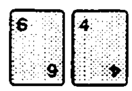


Form 75

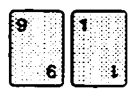


Form 84

Player 2



Form 64



Form 91

Each player creates two "wrestling teams" by writing each number as a sum of tens and ones.

	Player 1:	*Player 2:*
	75 * 84	64 * 91
Teams:	(70 + 5) * (80 + 4)	(60 + 4) * (90 + 1)

Next, each player's two wrestling teams wrestle each other in this way: Each member of the first team (for example, 70 and 5) is multiplied by each member of the second team (for example, 80 and 4). Then the four products are added.

Player 1:

(70 + 5) * (80 + 4)

(70 * 80) + (70 * 4) + (5 * 80) + (5 * 4) =

5600 + 280 + 400 + 20 = 6300

```
 5600
  280
  400
+  20
-----
 5000
 1200
  100
-----
 6300
```

Multiplication Wrestling (continued)

Player 2:

(60 + 4) * (90 + 1)

(60 * 90) + (60 * 1) + (4 * 90) + (4 * 1) =

5400 + 60 + 360 + 4 = 5824

$$\begin{array}{r} 5400 \\ 60 \\ 360 \\ +\quad 4 \\ \hline 5000 \\ 700 \\ 120 \\ 4 \\ \hline 5824 \end{array}$$

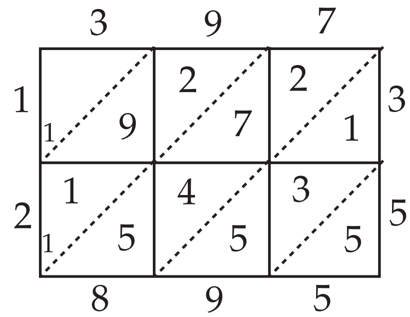
The player with the larger result wins the round. To find the winner's score, subtract the loser's result from the winner's result and record the difference on a score sheet like the one below. For example, Player 1 scores 476 points because 6300 – 5824 = 476. Player 2 scores no points.

Players' Names:	*Player 1*	*Player 1*
Round 1	*476*	*0*
Round 2		
Round 3		
Total		

In each round, each player forms two new numbers. Decide ahead of time on how many rounds or how long to play. At the end of the game, players add their scores. The player with the largest total wins.

Players may use a calculator to find the winner's score for a round and their total score for the game. They may also use a calculator to check a player's score for a round by multiplying the two numbers. (For example, to check Player 1's score, multiply 75 * 84.)

Lattice Multiplication



Partial Product Multiplication

```
  463
   x6
 ----
 2400
  360
 ----
   18
 2778
```

Low-Stress Division

```
7 ) 3 5 9
    - 7 0  |  1 0
    -----
    2 8 9
    - 7 0  |  1 0
    -----
    2 1 9
  - 1 4 0  |  2 0
    -----
    2 7 9
    - 7 0  |  1 0
    -----
        9
      - 7  |    1
    -----    -----
        2  |  5 1
```

Division Dash

1. Enter a number with at least three digits.

2. Press the square root key. If the display shows only 1 or 2 digits or if it ends in 0, start over. Use the final digit as the divisor and the 2 digits before it as the dividend.

3. Divide the dividend by the divisor. Try to calculate mentally. Write down the quotient and ignore the remainder.

4. Take turns repeating Steps 2 and 3. Keep a running total of your quotients. The first to 50 wins.

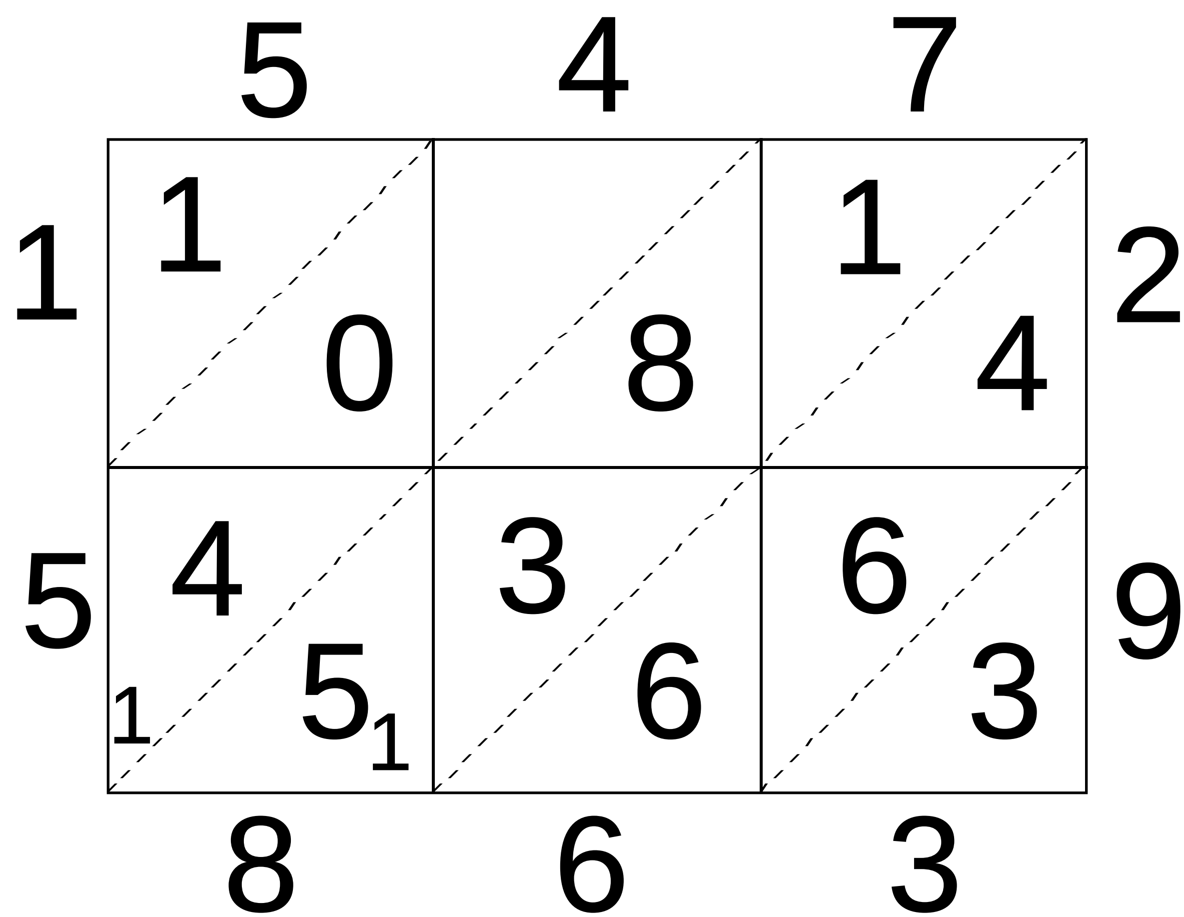
Questions about Multiplying & Dividing in *Everyday Mathematics*

1. How would you explain to parents the *Everyday Mathematics* use of invented and alternative algorithms instead of traditional long multiplication and long division?

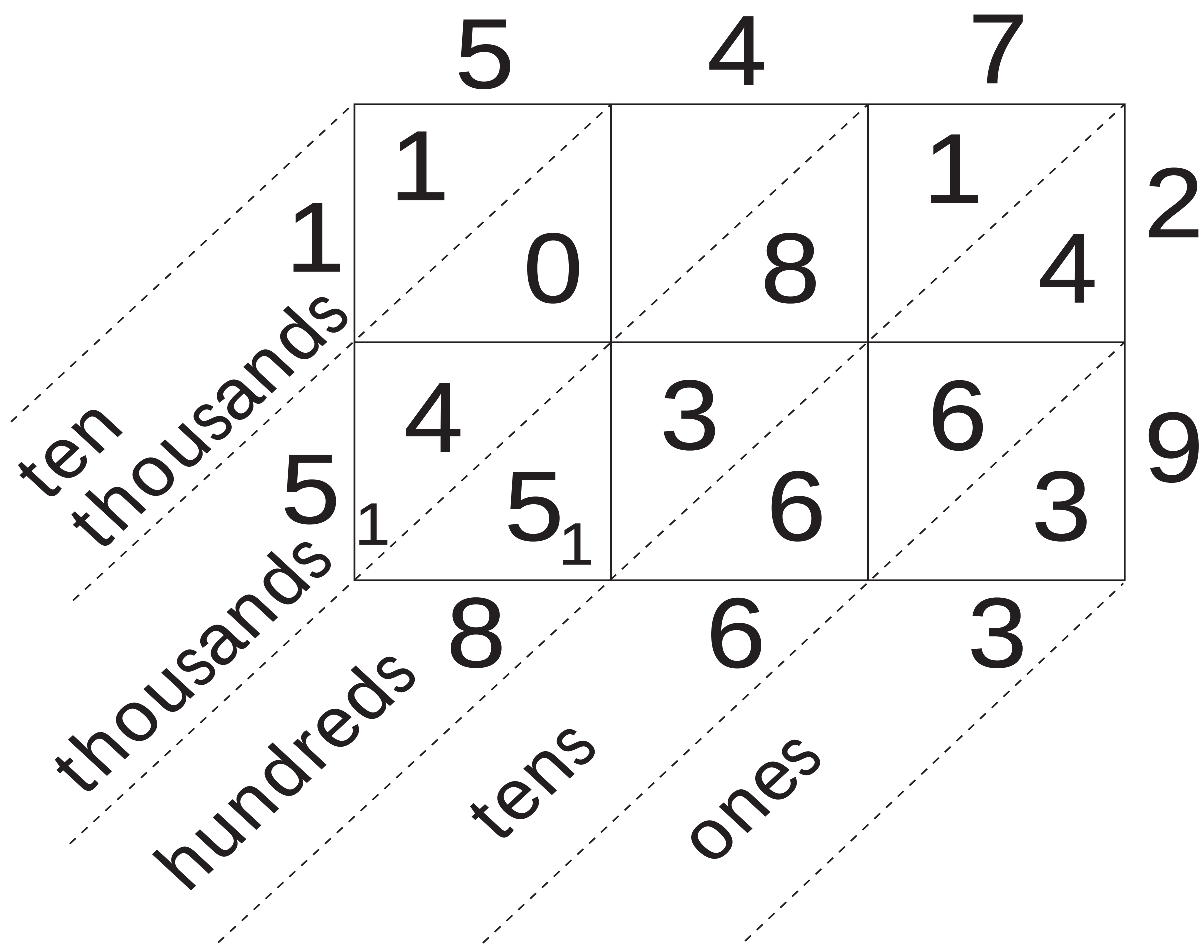
2. What do you think about the balance of mental arithmetic, paper-and-pencil skills, and calculators in *Everyday Mathematics*? How well are these balanced in the traditional curriculum?

3. What do you think about using games to practice computational skills?

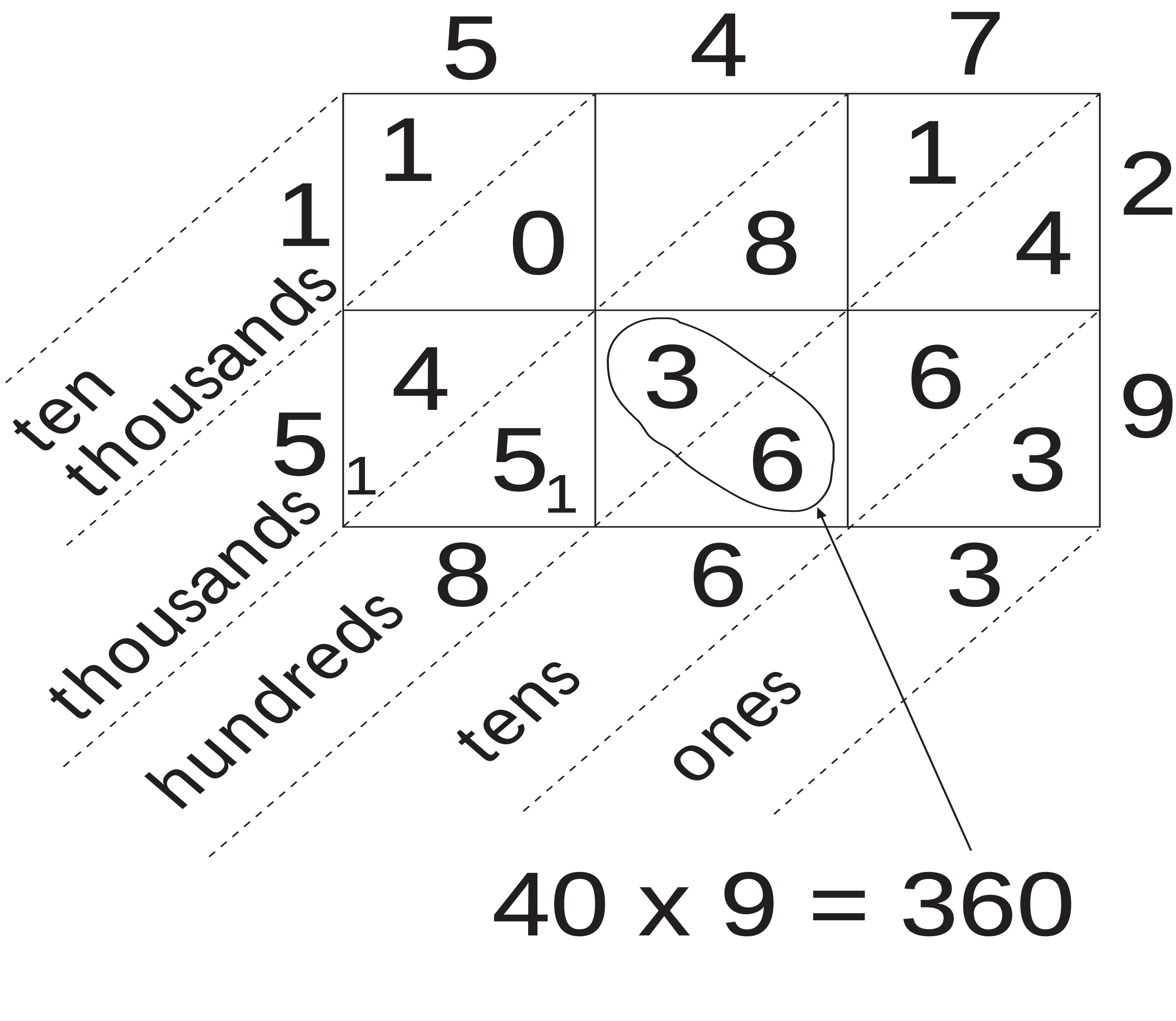
Lattice Multiplication



Place Value in Lattice Multiplication



One Cell's Value



Station A: Ancient Multiplication Algorithms — Activity 1

Lattice Multiplication

Materials:

Handout 3 (Lattice Multiplication)
Handout 4 (Blank Lattices)
Calculator

Tasks:

1. Do Handout 3.

2. Make up several problems and solve them with lattice multiplication. Check your work using another method. Use the blank lattices on Handout 4 or make your own.

3. Can you figure out why lattice multiplication works? Do you think lattice multiplication is more or less understandable than the traditional long multiplication algorithm?

4. What are some advantages of this method over the standard algorithm? What are some disadvantages?

5. Many *Everyday Mathematics* students prefer lattice multiplication to any other method. Why do you think this is so?

(Grades 3–5 activity)

Station A: Ancient Multiplication Algorithms **Activity 2**

Egyptian Multiplication

Tasks:

1. An ancient Egyptian multiplication algorithm required only addition and doubling.
   Here, in modern notation, are two examples, 29×36 and 18×74.

1	36	~~1~~	~~74~~
~~2~~	~~72~~	2	148
4	144	~~4~~	~~296~~
8	288	~~8~~	~~592~~
16	576	16	1184
29	1044	18	1332

2. Here's how to use this method.
   a. List the powers of 2 up to the largest power of 2 that is less than or equal to the first factor.
   b. In the second column, write the second factor next to the 1, and double it repeatedly for each factor of 2 in the first column.
   c. Check off the powers of 2 that add to the first factor. Add them to be sure.
   d. Cross out the rows that are not checked off.
   e. Add the numbers in the second column that are not crossed out. The sum is the answer.

3. Make up several problems and solve them with Egyptian multiplication. Check your work using another method.

4. Can you explain why this method works?
   Hint: $29 \times 36 = (1+4+8+16) \times 36 = (1 \times 36) + (4 \times 36) + (8 \times 36) + (16 \times 36)$.

Station B: Arrays & Partial Products **Activity 1**

Array Multiplication

Materials:

Handout 5 (Array Grid)
Handout 6 (Array Record Sheet)
Base-10 Pieces
Green, red, and blue colored markers, pencils, or crayons

Tasks:

1. Cover a 12-by-14 array of squares on the array grid with as few base-10 blocks as you can.
   Start at the lower left corner.
   Use flats first, then longs, and cm cubes last.

2. Make a picture of your 12-by-14 array on the Array Record Sheet.
   Color the squares covered by flats green.
   Color the squares covered by longs red.
   Color the squares covered by cm cubes blue.

3. Record the results next to the picture.

4. Color the bottom array on the Array Record Sheet to show how a 21-by-24 array could be covered with as few base-10 blocks as possible.

(Grade 3 activity)

Station B: Arrays & Partial Products **Activity 2**

Partial Product Multiplication

Tasks:

1. Here are two examples of the partial product multiplication algorithm.

463	57
x 6	x 89
2400	4000
360	560
18	450
2778	63
	5073

2. Solve 12 x 14 and 21 x 24 by partial product multiplication.
   Note how the partial products correspond with the colored regions in Array Multiplication.

3. What are some advantages of this method over the standard algorithm?
   What are some disadvantages?

4. Compared to students in other programs, *Everyday Mathematics* students are better able to solve problems like 70 x 8 and 5 x 400. Why is that ability important? How might partial-product multiplication contribute to developing that ability?

PROCEDURES FOR MULTIPLICATION AND DIVISION IN *EVERYDAY MATHEMATICS*

Station C: Division **Activity 1**

Low-Stress Division

Tasks:

1. Here are two examples of a low-stress division algorithm, each showing that 359 ÷ 7 is 51, remainder 2.

```
7)3 5 9 |
 -3 5 0 | 5 0
 ------ |
      9 |
     -7 |    1
     -- | ----
      2 | 5 1
```

```
7)3 5 9 |
   -7 0 | 1 0
  ----- |
  2 8 9 |
   -7 0 | 1 0
  ----- |
  2 1 9 |
 -1 4 0 | 2 0
 ------ |
    7 9 |
   -7 0 | 1 0
 ------ |
      9 |
     -7 |    1
 ------ | ----
      2 | 5 1
```

2. Make up several problems and solve them with this low-stress division algorithm. Check your work using another method.

3. Why does this method work? What advantages does it have over the traditional algorithm? What disadvantages?

(Grades 4–5 activity)

PROCEDURES FOR MULTIPLICATION AND DIVISION IN *EVERYDAY MATHEMATICS*

Station C: Division **Activity 2**

Getting to One

Materials

Calculators (2)
Handout 7 (Rules for Getting to One)

Tasks

1. Read Handout 7 and play a round or two.

2. What useful division skills does this game involve?

(Grades 4–5 activity)

Station D: Multiplication Wrestling **Activity 1**

Materials:

Calculator
Deck of number cards
Handouts 8 and 9 ("Multiplication Wrestling" directions)

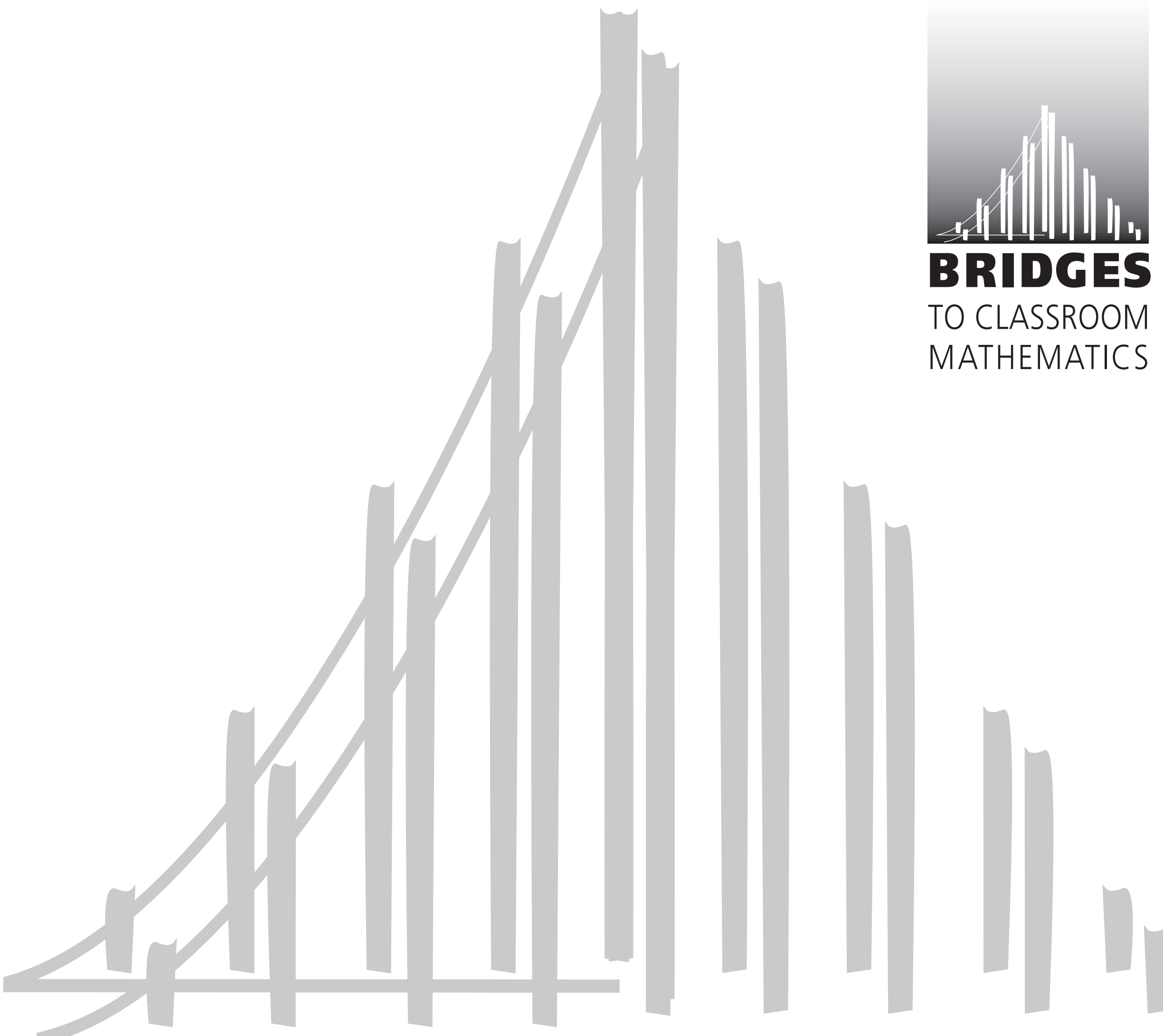
Tasks:

1. Read the game directions on Handouts 9 and 10 and play several rounds.

2. With four different number cards, there are 12 different multiplication problems that can be made. Can you find a strategy that helps you choose the problem with the biggest product?

(Grades 4–5 activity)

S A M P L E R



BRIDGES
TO CLASSROOM
MATHEMATICS

An *Investigations in Number, Data, and Space* Unit

The Role of Games in *Investigations in Number, Data, and Space*

Session Overview:

Activity (Name)	Description	Time	Materials
Games and Math	Participants examine games they and their students have played, such as board and card games, and discuss the mathematical thinking that is involved.	15 - 20 minutes	Transparency 1: Where's the Mathematics? Transparency 2: The Purposes of Games in Investigations Chart paper or board Markers
Game play and analysis	Participants learn and play a few Investigations games, and analyze mathematical understandings gleaned from playing these games	60 - 75 minutes	Transparency 3: Guidelines for Playing the Games Handout 1: Reflection Sheet for Games Game packets
Discussion of findings	Two games are discussed with the entire group: Close to 100 and one game of the instructor's choosing.	20 - 30 minutes	Handout 2: A Selection of K-2 Number Games from *Investigations* Chart paper Markers Tape

Summary of Session:

In this session participants work with number games from different grade levels of *Investigations*. They play the games with an eye toward understanding the underlying mathematics, the ways in which teachers "talk mathematics" with children as game play proceeds, and ways in which they might adapt games to challenge children at different levels. Participants discuss the mathematical learning that takes place through Investigations games.

Goals:

- To learn about the role of games in the *Investigations* curriculum and how these games can support students' learning at different levels throughout the year;
- To understand how games are used as an opportunity to practice computation;
- To learn how games serve as a link between home and school learning;
- To learn how games can be used to better understand students' mathematical thinking, and to set goals for students;
- To consider ways that games can be adapted to meet the needs of individual students.

Planning Ahead for the Session:

Play each of the games (Double Compare, Close to 100/1000, Fraction Cookie Game; Multiple and Factor Bingo) ahead of time to become familiar with them and with the underlying mathematics. Consider what personal experiences you have to share from working with students when you discuss Transparency #1: The Purposes of Games in Investigations.

Figure out logistics for participants to play the games in your workshop. Pairs or groups of four will play the games. Everyone will need to play "Close to 100" plus one other game of your choosing.

Zip lock bags work well to hold the game instructions and materials in each packet. There are four different game packets (see below). Decide how many of each packet you'll need , based on the number of participants you have (read below for guidance). If you will be doing this workshop frequently, it is helpful to prepare directions for games and some of the game boards on card stock and/or laminated surface for reuse.

Familiarize yourself with all transparencies and handouts. Make the appropriate number of copies of each handout.

Materials needed for each game packet:

Each game packet is planned for four people playing at a time.
(For Close to 100 and the other game everyone will play, you will need about 1 packet per 6 people; for the other games you will need 1 packet per 8-10 people. Black line masters for sheets for these game packets are included at the end of this document.)

Double Compare: (in *Mathematical Thinking at Grade 1* and other units grades K-1)

- Directions
- 2 decks of primary number cards or numeral cards

Fraction Cookie Game: (in *Fair Shares*., grade 3, useful for grades 3-5)

- Directions
- 3 fraction cubes (dice)--2 of one color, one of another color, each with 1/2 1/2 1/3 2/3 1/6 5/6
- 4 Hexagon Sheets (can be laminated)
- 1-2 buckets of pattern blocks for the entire group to share, or 4-5 handfuls of green triangles, blue rhombuses, red trapezoids, and yellow hexagons

Close to 100/1000: (in *Landmarks in the Thousands*., grade 4, other units grades 3-5)

- Directions
- 2 decks of numeral cards
- Score Sheet

Multiple and Factor Bingo: (from Ten Minute Math, *Picturing Polygons* , grade 5)

- Directions
- 4 laminated 100 charts (or several paper ones)
- 4 laminated multiplication tables (or several paper ones)
- 1 deck of Multiple Bingo Cards
- 1 deck of Factor Bingo Cards
- 20 bingo chips (if using laminated 100 charts), or colored pens.

Other Materials

Chart paper, markers, and masking tape
Transparency 1: Where's the Mathematics?
Transparency 2: The Purposes of Games in *Investigations*
Transparency 3: Guidelines for Playing the Games
Handout 1: Reflection Sheet for Games, 2 per person
Handout 2: A Selection of K-2 Number Games from *Investigations*

Activity 1 – Games and Math

Activity Synopsis:
Participants examine games they and their students have played, such as board and card games, to determine the mathematical thinking involved. There is a brief discussion of the role of games in Investigations, with emphasis on the importance of games in practicing computation and on the role of games in linking math at home with math at school.

Materials:
- Transparency 1: Where's the Mathematics?
- Transparency 2: The Purposes of Games in Investigations
- Chart paper or (white) board
- Markers

Time:
15 -20 minutes

Introduction to the Activity:
Tell the group that in this session, you will be looking at the role of mathematical games in Investigations. The focus will be on finding the important mathematics that underlies each game. To begin, you will talk briefly about the mathematics in familiar, everyday games.

Conducting the Activity:

You might begin by saying something like:

We're going to brainstorm a list of some board and card games you and your families have played--games that involve some work with numbers and some which involve other mathematics.

Ask people to think for a moment about these everyday games and make a list of them on the board or on chart paper. Encourage participants to consider games for early elementary school children and for older children, games for geometry and logical thinking as well as for number practice.

After you have games on the list,

display Transparency 1: Where's the Mathematics?.

Ask participants in small groups to choose a few of the games listed or other games and discuss these questions:

• What mathematical ideas or understandings does this game promote?
• What mathematics is involved in effective strategies for playing this game?
• What numerical understanding is involved in scoring this game?
• How much of this game involves mathematical skill versus luck?

Don't spend much time on this activity. The idea is to get people involved in thinking about familiar games before they proceed to examine the games in Investigations.

You might finish up this part of the activity by asking the whole group what games they would recommend for primary grade children. What games would they recommend for older elementary school children? Are there any logic games or geometric/ spatial games for early elementary children?

Games in Investigations

To introduce the role of games in Investigations, say something like the following to the group:

In Investigations, students play mathematical games a great deal. The games serve several important purposes.

Display Transparency 2: The Purposes of Games in Investigations, and review it with the group.

The Transparency lists the following ideas:

• Games are a central part of the mathematics in the units, not just enrichment.

• Games develop familiarity with the number system and with "landmarks" in the number system, such as 10s, 100s, and 1000s, and provide engaging opportunities for practicing computation.

• Playing games encourages strategic mathematical thinking and demands that students find an optimal way (rather than just any way) of solving a problem.

• Games are played often throughout a unit and throughout the year to develop fluency with numbers. It's expected that students will play a game many times.

• Games provide a school to home link. Parents learn about the mathematical thinking their children are doing by playing games with them at home.

Highlight main points on the transparency.

Stress the first point, that games in Investigations are integral activities used to develop mathematical understanding and for practice. They are not just isolated activities, but are preceded and followed by related activities. Talk about your own perspective on the value and uses of Investigations games.

Activity 2 – Playing and Examining Investigations Games

Activity Synopsis:

This activity focuses on learning and playing a few Investigations games, and analyzing the mathematical understandings that are developed while playing these games. Participants play the games in small groups and answer questions about the mathematics in the game, the teacher's role in promoting student thinking during games, and making adaptations in the games in order to challenge students appropriately.

Materials:

- Transparency 3: Guide to Playing the Games
- Handout 1: Reflection Sheet for Games (4 per group--can be 2, printed back and front)
- Game packets as described on pages 2-3 in detail.

Time:

1 hour - 1 hour 15 minutes (Allow 30 minutes for discussion at the end of the session.)

Conducting the Activity

Explain that participants will work in groups of four to play Investigations games. These games can be adapted for use with students at different levels or slightly different grades. The games involve work with whole number computation and fractions. The four games are:

- Double Compare (from Kindergarten and Grade 1 number units; useful also in grade 2)
- Close to 100/Close to 1000 (from third, fourth, and fifth grade number units)
- The Fraction Cookie Game (From Third grade *Fair Shares)*
- Multiple Bingo (fourth and fifth grade Ten Minute Math--can be found in *Picturing Polygons*)

Choose one of the games to start with. Read the instructions carefully and then spend about 5-10 minutes playing the game. As you play, think about the mathematics involved. When you are finished playing, discuss the questions on the "Reflection Sheet" with your partner(s) and jot down your responses. Then, choose another game and repeat the same process. (The reflection sheets are just for your thinking and for the group discussion. We will not collect them.) You should have time to work with at least three of the games. EVERYONE NEEDS TO PLAY CLOSE TO 100/1000 AND ________ (the other game you've selected.) **We will discuss these two games in the last part of the session.**

Answer any questions about this process, and explain how materials will be handled. (You may want to distribute games and Reflection Sheets to tables, or you may want to arrange them in a central place for participants to pick up on their own and put back when they are finished.)

Display Transparency 3: Guidelines for Playing the Games, and talk a bit about each game.

Point out that the game packets have materials for four people to play. Groups of 3 is fine also. Instructions for the games are included in each packet. The guidelines on the overhead are for them as adults becoming familiar with the games.

Distribute one or two game packages and 4 copies (or 2 copies back to front) of Handout 1: Reflection Sheets for Games to each group of 3-4 participants.

As participants are working on the games, check in with each group to do the following:

- Help clarify any ambiguities in the instructions, and make sure people understand the instructions.
- Observe game play and ask questions about participants' strategies. Try to model good question-asking.
- Encourage participants to think hard about the important mathematical ideas in each game.
- Note participants' response to questions on the Reflection Sheet, so you can bring up salient points during the group discussion which follows this activity.
- Make sure people don't get bogged down in game play: They need time to do the analysis.

Some suggestions for questions and specifics to be aware of for each game:

<u>Double compare</u>: Ask how they figure out which sum is greater without actually finding the sum. Ask how they might modify the game to make it easier or harder. Might they use it with third grade?

<u>Fraction Cookie Game</u>: In the activity in the third grade unit, Fair Shares, before this game is introduced, students cover the hexagons shapes on paper with the pattern blocks in as many different ways as they can, draw in the shapes and label each shape with the fraction. Then they write the fractions from each way they covered it as an equation, e.g. 1/2 + 1/3 + 1/6 = 1; 1/3 + 1/3 + 1/6 + 1/6 = 1 (or 2/3 + 3/6 = 1). The fraction cookie game would be useful in the fourth and fifth grades as well.

Close to 100/1000: It is likely that some of the participants will have played these games in workshops or with their class. Encourage those who are familiar with the game to think about good questions to ask students. Teachers have noticed that a number of students who are introduced to this game after they learn column addition will make 2 digit numbers randomly and add them, then make another pair and add them. To encourage them to plan ahead what numbers they want, make one 2-digit number from their card and then ask them about how large they would like the other number to be to have a sum of 100.

Multiple Bingo/Factor Bingo: The multiple Bingo game is a good game for fourth and fifth grade students to play. Factor Bingo is much more difficult and may be appropriate for only a few students or for older students. As you help participants get started, note that the need to have the 100 board side of the card turned up to play Multiple Bingo. They will use the multiplication table to play Factor Bingo. Notice and mention any reference participants make to patterns of multiples on the 100 board, such as "You can find the multiples of 9 by looking at this diagonal". When they play Factor Bingo, notice different ways participants use to find factors--factor trees, calculator, factor pairs--and encourage them to show one another the strategies they are using.

A few minutes before you call the group together, remind participants that the next part of the session will involve a discussion of the questions on the Reflection Sheet for Close to 100 and (one other game), so that they can think about what they want to say.

Activity 3 – Discussing Investigations Games

Activity Synopsis:
Using Close to 100/1000 and one other game of your choosing as focus games, the entire group discusses the questions on the Reflection Sheet.

Materials:
- Handout 2: A Selection of K-2 Number Games from *Investigations*
- Chart paper
- Markers
- Tape

Time:
20 - 30 minutes

Conducting the Activity
This is a time to delve into several aspects of Investigations teaching and learning, as embodied in the games. The structure of this activity is simple: You spend 10 to 15 minutes discussing the questions on the Reflection Sheet with reference to Close to 100/1000 (which everyone has played). Start by asking about the mathematical ideas in the games but then accept any responses to questions on the sheet. Write participants' responses on chart paper. Post each piece of chart paper where everyone can see it.

Repeat this process for the second game you have chosen, again allowing about 10 to 15 minutes for the discussion.

Post the chart papers.

Take some time at the end for participants to think about what they learned in this session:

What did you learn in this session
about mathematical ideas?
about using games?
about children's thinking?

Encourage responses about mathematics learned to include smaller ideas such as the fact that $1/3 + 1/6 = 1/2$, as well as larger ideas such as using a factor tree to find factors.

Because most of the games participants played in this session are for older elementary school students, we include a handout with some suggestions for games for Kindergarten to Grade 2.

Distribute Handout 2, A Selection of K-2 Number Games from *Investigations.*

In keeping with the spirit of the "Investigations" games, you might make the game packets available for play during lunch time and before and after the workshop sessions. If you're doing a workshop that lasts more than one day, you may want to make the packets available for checking out overnight.

For your information, listed below are ideas that teachers have suggested in response to questions on the Reflection Sheet with respect to Close to 100/1000 and the Multiple and Factor Bingo Games.

Close to 100/1000

Mathematical understanding

- seeing the relationships between tens and ones (and hundreds)
- developing and practicing addition strategies for quantities to 100 (or 1000)
- comparing several possible solutions to find closest to 100 or 1000.
- combining subtraction and addition (or addition of positive and negative numbers) when playing the scoring variation.

Questions to ask

- I see you're starting with _____. What's your plan?
- Does it work better to work with the larger numbers (tens or hundreds) first or the smaller numbers (ones) first? Why?
- What could you do with your wild card to get even closer?
- Do you think you've gotten as close as you possibly can? How do you know?
- Do you think it's usually possible to hit 100 exactly? Can you always do it if you have a wild card? If you have 2 wild cards?
- For scoring variation with positive and negative numbers: Is this game easier or harder to play than the regular version? Why?

How to Modify

Note: Often, participants assume this game will be too easy for their students. Try the game as written with all students and learn how they handle it before making modifications.

- Play Close to 10,000, dealing out 10 cards and using 8 each time.
- Play close to 10, dealing out 4 cards and using the two with a sum closest to 10.
- Provide decimal point cards to use as needed, deal out 6-8 cards and use any number of the cards to play Close to One.

Management and Other Ideas

- Allow students to choose whether to play cooperatively or competitively.
- Sometimes students should record their equations, but other times they may just want to keep track of overall score.
- This is a good game to play at "Math Night" for families.
- This is a good game to use for individual assessment. Write a "hand" of digits on the board and have children determine their best solution and explain why it's closest to 100.
- You can use regular playing cards, without the face cards, to play this game.
- Have a bunch of games accessible for play during recess, before school, etc.
- Encourage children to make their own set of numeral cards to bring home.
- Teaching this game to someone else often helps children explain their own effective number strategies.

Multiple and Factor Bingo Games

Mathematical Understanding

- Flexible thinking about factors
- Reinforcement of multiplication facts
- Recognition of factors and divisibility rules
- Knowledge of prime numbers
- Seeing patterns of multiples on the 100 board
- Knowing the difference between factors and multiples
- Skip counting with different steps - distance between steps
- Developing a visual strategy for winning

Questions To Ask

- Prove to me that your number is a multiple (or factor) of the number picked.
- What other numbers could you have covered? Could you use a higher or lower number?
- How did you choose a number for a wild card?
- What strategies can you share with classmates?
- What could you do to change the game?
- Ask why certain numbers are harder/easier.

How To Modify

- Let students write all the factors on their factor Bingo cards.
- Limit factor cards to a few families or increase the factor families.
- Use more smaller multiples (30, 24, 42) or more big multiples.
- Play multiple Bingo with a 200 chart or 300 chart.
- Use calculator to figure out factors and multiples.
- Disallow first row and multiples of 10.

Management and Other Ideas

- Laminate 100s boards and reuse.
- When teaching the game, brainstorm all the possibilities
- Group by pairs of similar ability levels, one pair versus another pair
- Play collaboratively
- Kids find multiples easier than factors.

Reflection Sheet for Games

Name of Game____

1. What are the important mathematical ideas and understandings that this game promotes? What strategies make use of the mathematics?

2. What questions might you ask students during or after the game? What information would this give you about the student?

3. How would you modify this game to meet the range of students in your classroom?

4. What can you share with other teachers about how you would (or have) used this game with your class? (management issues, logistics, things that surprised you, etc.)

A Selection of K-2 Number Games from *Investigations in Number, Data, and Space*

Games are used throughout Investigations to support students in developing and practicing important concepts and skills in number, geometry, and data. Many of the number games involve making sums that equal (or come close to) important "landmarks" in our number system, such as 10, 100, and 1000. The games described here for grades K-2 all use the Primary set of number cards that contain numbers and pictures. Some of the games include variations that you can use to adjust the level of challenge; you might decide to create your own variations. As you observe your students playing, think about what students are learning and ways that they are demonstrating their mathematical understandings.

- <u>Tens Go Fish</u> The object of Tens Go Fish is to find pairs of cards that total 10 (such as, 6 and 4, or 7 and 3). Students begin with 5 cards each, and put down pairs totaling 10 and replace those cards. Then the game is played by taking turns requesting a certain number and drawing a card from the deck if the number is not available. This game is included in Grades 1 and 2 (Grade 1: *Number Games and Story Problems*; Grade 2: *Mathematical Thinking at Grade 2*, and *Coins, Coupons, and Combinations*).

- <u>Total of 10</u> A deck of Number Cards is laid out face up in rows of 5. Students take turns finding combinations of cards that total 10 (such as, 5, 2, and 3, or 6, 2, 1, and 1). At the end they write down all the combinations they made using addition notation. This game is included in Grade 1 (*Number Games and Story Problems*), and a very similar game, Turn Over 10, played like Concentration, is include in Grade 2 (*Mathematical Thinking at Grade 2*, and *Coins, Coupons, and Combinations*).

- <u>Close to 20</u> (Grade 1: *Number Games and Story Problems* (extension), Grade 2: *Coins, Coupons, and Combinations*, page 143 and 144) This game is a variation of Close to 100. It uses the K-2 number cards and interlocking cubes. From 5 cards, players each chooses three cards that total as close to 20 as possible. Their score for the round is the difference between the total and 20, and they take that number of cubes. The winner is the player with the fewest cubes (the lowest score) after five rounds.

- <u>Collect 25¢ Together</u> (Grade 1: *Number Games and Story Problems*, page 64; and Grade 2: *Coins, Coupons, and Combinations*, page 80) In this game, pairs take turns throwing 2 dice or turning over a number card and collecting coins with that amount in cents. The goal is to cooperatively collect 25 cents or just over. For more challenge, students collect a larger amount such as 50 cents.

WHERE'S THE MATHEMATICS?

What mathematical ideas or understandings does this game promote?

What mathematics is involved in effective strategies for <u>playing</u> this game?

What numerical understanding is involved in <u>scoring</u> this game?

How much of this game involves <u>mathematical skill versus luck</u>?

Purposes of Games in Investigations

• **Games are a central part of the mathematics in the units, not just enrichment.**

• **Games develop familiarity with the number system and with "landmarks" in the number system, such as 10s, 100s, and 1000s, and provide engaging opportunities for practicing computation.**

• **Playing games encourages strategic mathematical thinking and demands that students find an optimal way (rather than just any way) of solving a problem.**

• **Games are played often throughout a unit and throughout the year to develop fluency with numbers. It's expected that students will play a game many times.**

• **Games provide a school to home link. Parents learn about the mathematical thinking their children are doing by playing games with them at home.**

Guidelines for Playing the Games

<u>Double Compare</u>: Grades K-2. Best played by two people; each pair splits one pack. Look especially for ways to know which sum is larger without actually computing the sum.

<u>Close to 100/Close to 1000</u>: Grades 3-5, with variations for younger grades. Start with Close to 100; also try Close to 1000. Pairs can play cooperatively. Take turns taking the lead.

<u>Fraction Cookie Game</u>: Grades 3-5, Two to four people play, each with their own hexagon board. Read about "Trading up" in the beginning instructions. Start play with the intermediate game: adding fractions on 2 dice. Try the advanced variation as well. Observe different ways of making exchanges. What fraction facts are you using?

<u>Multiple and Factor Bingo</u>: Grade 4 and up. Play Multiple Bingo first. Each player needs a 100 chart and bingo chips or a dry-erase marker (for laminated chart) to mark numbers. Try Factor Bingo as well; this is played on the multiplication table (on other side of 100 chart). It is very challenging; feel free to use a calculator.

Materials for Game Packets: (18 pages)

Materials needed for <u>each game packet:</u>

(For Close to 100 and the other game everyone will play, you will need about 1 packet per 6 people; for the other games you will need 1 packet per 8-10 people. Black line masters for sheets for these game packets are included at the end of this document.)

Double Compare: (from K and 1 including *Mathematical Thinking at Grade 1*)
- Directions
- 2 decks of primary number cards or numeral cards *

Fraction cookie game: (from *Fair Shares*., grade 3)
- Directions
- 3 fraction cubes (dice)--2 of one color, one of another color, each with 1/2 1/2 1/3 2/3 1/6 5/6,
- 4 Hexagon Sheets (can be laminated)
- 1-2 buckets of pattern blocks for the entire group to share, or 3-4 handfuls of green triangles, blue rhombuses, red trapezoids, and yellow hexagons

Close to 100/1000: (from 3, 4, 5 including Landmarks *in the Thousands*., grade 4)
- Directions
- 2 decks of numeral cards
- Score Sheet

Multiple and Factor Bingo: (from Ten Minute Math, *Picturing Polygons* , grade 5)
- Directions
- 4 laminated 100 charts (or several paper ones)
- 4 laminated multiplication tables (or several paper ones)
- 1 deck of Multiple Bingo Cards
- 1 deck of Factor Bingo Cards
- 20 bingo chips (if using laminated 100 charts), or colored pens.

* Note: Black line master for Primary Number Cards is not included here. You can use ready made cards, numeral cards from black line masters provided below for close to 100, or copy numeral cards from a grade K, 1, or 2 number unit.

Double Compare Instructions

Materials: Deck of Number Cards (4 each of 0–10) (remove the Wild Cards)

Players: 2

Object: Decide which of two sums is greater.

> **Note to families:**
> In this game, your child will be finding the totals of pairs of numbers. You will need a set of Number Cards to play this game.

How to Play

1. Mix the cards and deal them evenly to each player. Place your stack of cards face down in front of you.
2. At the same time, both of you turn over the top two cards in your stack. Compare your cards to your partners to determine which sum is more. If your total is more than the other player's, say "Me!" If the two totals are the same, turn over the next two cards and compare these sums.

   Sometimes you may be able to decide which pair is more without actually figuring out the total.
3. Keep turning over two cards. Say "Me!" each time your total is more.
4. The game is over when you have both turned over all the cards in your stack.

Variations

- Remove the 7-10 cards from the deck, and play with just the 0-6 cards.
- Play Compare. Players turn over one card on a turn. The player with the larger number says "Me!"
- Add the four wild cards to the deck. A wild card may be used as any number. Challenge students to use it for the lowest number that will allow them to win.
- Play Triple Compare. Players turn over three cards on a turn. The player with the larger total says "Me!."

Fraction Cookie Game Instructions

Basic Activity

Students play this game in pairs. The object is to be the first to collect or give away a given number of hexagon "cookies." The fraction die or Fraction Card tells you what to add to (or subtract from) your cookie.

The Fraction Cookie game provides practice with recognizing and visualizing common fraction combinations based on sixths, such as one third plus one sixth equals one half. Students focus on:

- Identifying fractional parts
- Exchanging equivalent fractions
- Adding and subtracting fractions
- Relating numerical fractions to equivalent visual representations

Materials

- Fraction dice or Fraction Cards (in two colors) (1 per pair)
- Pattern blocks
- Copies of the Hexagon Cookie Sheet
- Colored pencils, markers, or crayons

Procedure

Beginning Game (One Die)

Each pair of students should have a set of pattern blocks, a fraction die or deck of Fraction Cards, copies of the Hexagon Cookie Sheet, and something to color with. Pairs put the pattern blocks in a pile between them. Players take turns rolling the die or drawing a card and picking an equivalent pattern block to add to their cookie. For example, if a student rolls 1/3, the player takes a blue diamond (1/3 of a yellow cookie). Before beginning, students decide how many cookies are needed to win the game.

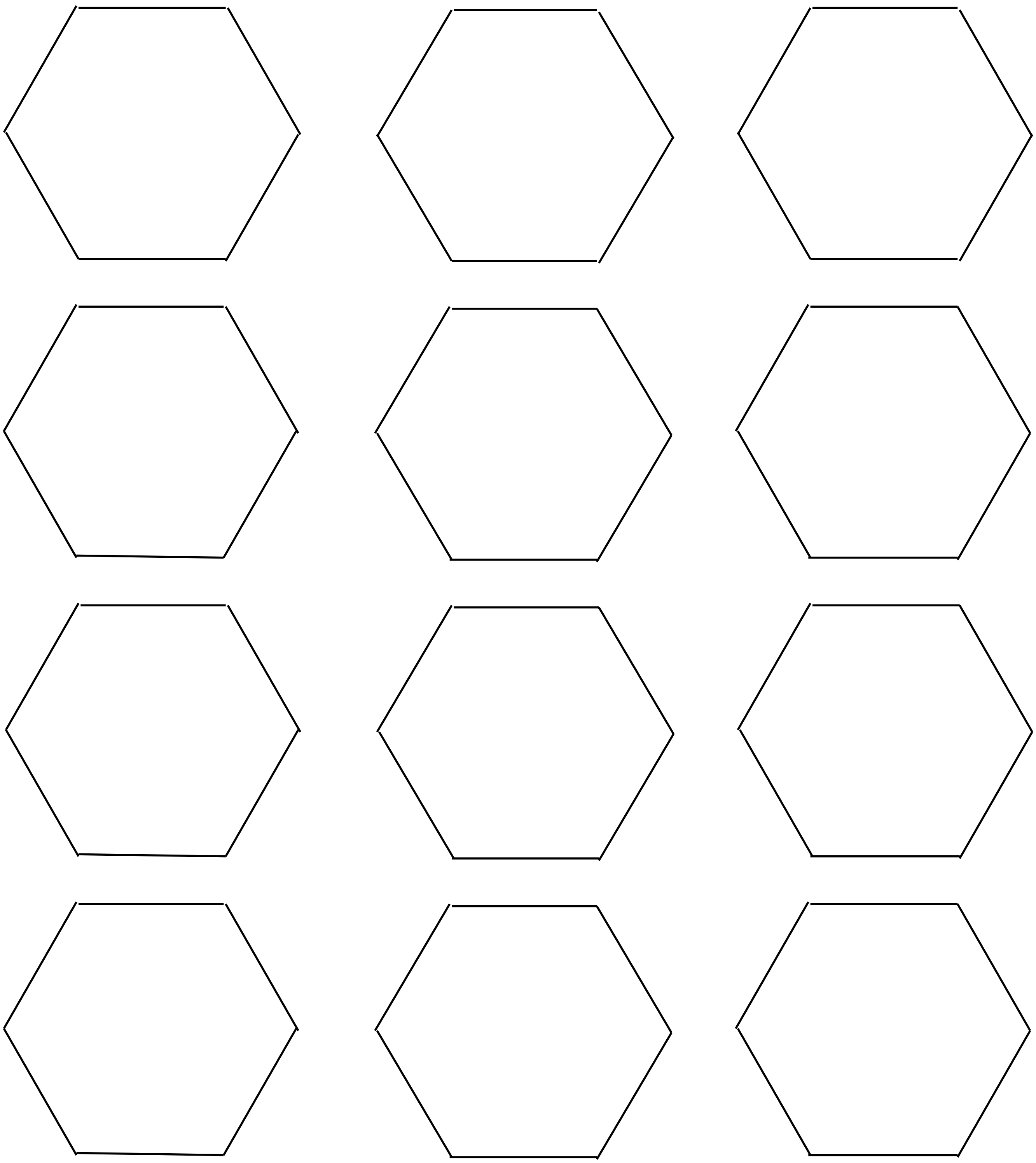
"Trading up" is a basic part of the game. Players must find combinations of two or more smaller blocks and exchange them for single, larger pieces which are equal to those combinations. In this way, students should have the least amount of pattern blocks possible at the end of each turn. For example, a player with 2 1/2 cookies should have 2 yellow and one red (1/2) pattern block.
After each round, students check each other's work.

Note: Players can 'build' their cookies on top of the Hexagon Cookie Sheet, and remove them when a whole cookie is completed. These sheets are also useful if certain blocks (particularly yellow hexagons) are in short supply, as students can color them in to record their completed cookies.

Fraction Cookie Game Instructions (cont.)

Variations
Intermediate Game: Adding Fractions (Two Dice) As students seem ready, increase the level of difficulty by adding another fraction die to the game. Students now throw two dice or draw two cards, and must add the fractions to determine how much cookie to take. As in the basic game, students finish their turn by "trading up" to end with the fewest possible pieces, and by checking each other's work.
Advanced Game: Adding and Subtracting (Three Dice) When students are comfortable with the Intermediate level, introduce a third die of a different color (or a second deck of cards of a different color). Players now roll three dice and add the amounts on the same-color dice, and then subtract the fraction on the die of the other color. (Students playing with cards draw two cards of the same color, and one card of the other color, and follow the same instructions.) They then add or subtract the result to their cookies. In this version, students start with two hexagon cookies so they won't run out when they subtract. Again, students "trade up" and check each other's work at the end of each turn.
Subtracting Each student begins with three whole yellow hexagons or cookies. Players subtract the amount they roll from their cookies. The goal is to be the first player with no cookies left. Players must finish with the exact fraction (for example, a player with one-sixth of a cookie left cannot remove it when one-half is rolled or drawn; they must roll a one-sixth). Players may decide to play with only one die or with two dice throughout or each player may choose whether they roll one die or two for each turn throughout the game.

Hexagon Cookie Sheet for Fraction Cookie Game



How to Play Close to 100

Materials

- One deck of Numeral Cards
- Close to 100 Score Sheet for each player

Players: 1, 2, or 3

How to Play

1. Deal out six numeral cards to each player.
2. Use any four cards to make two numbers. For example, a 6 and a 5 could make 65 or 56. Wild cards can be used as any numeral. Wild cards can be used as any numeral. Try to make numbers that, when added, give you a total that is close to 100.
3. Write these numbers and their total on the Close to 100 Score Sheet. For example: 42 + 56 = 98.
4. Find your score. Your score is the difference between your total and 100.
5. Put the cards you used in a discard pile. Keep the two cards you didn't use for the next round.
6. For the next round, deal four new cards to each player. Make more numbers that come close to 100. When you run out of cards, mix up the discard pile and use them again.
7. After five rounds, total your scores. Lowest score wins.

Variations

Alternate Scoring

Write the score with plus and minus signs to show the direction of your total away from 100. For example: If your total is 98, your score is -2. If your total is 105, your score is +5. The total of these two scores would be +3. Your goal is to get a total score for five rounds that is close to 0.

Close to 1000

two numbers that total as close as possible to 1000. For example, if you are dealt 4, 5, 8, 3, 2, 9, 9, and 0, you might try 420 + 583 (1003). The score for each round is the difference of the total and 1000. In the example above, the score is 3. Play five rounds. The total lowest score wins. Again, as students become comfortable with this version, negative and positive integers may be introduced as a scoring variation.

Close to 0 with Two or Three Digit Numbers

To play with 2-digit numbers, deal out 6 cards to each player. Each player uses any four cards to make two numbers whose difference is as close as possible to 0. For 3-digit numbers, deal out 8 cards to each player. Each player uses any six cards to make two numbers whose difference is as close as possible to 0. With the numbers above, a player might try 402 - 399. Three is 3 away from 0, so the player's scored would be 3. Play five rounds and the lowest total score wins.

Numeral cards for Close to 100/1000 (can be used for Double Compare)

Numeral Cards A (1 of 3 pages)

0	0	1	1
0	0	1	1
2	2	3	3
2	2	3	3

Numeral Cards B

4	4	5	5
4	4	5	5
6	6	7	7
6	6	7	7

Numeral Cards C

8	8	9	9
8	8	9	9
Wild Card	Wild Card		
Wild Card	Wild Card		

Close to 100 Score Sheet

Name: Score:

Round 1: _______ _______ + _______ _______ = _____________ ___________

Round 2: _______ _______ + _______ _______ = _____________ ___________

Round 3: _______ _______ + _______ _______ = _____________ ___________

Round 4: _______ _______ + _______ _______ = _____________ ___________

Round 5: _______ _______ + _______ _______ = _____________ ___________

TOTAL SCORE: ___________

Close to 1000 Score Sheet

Name: Score:

Round 1: _____ _____ _____ + _____ _____ _____ = _____________ ___________

Round 2: _____ _____ _____ + _____ _____ _____ = _____________ ___________

Round 3: _____ _____ _____ + _____ _____ _____ = _____________ ___________

Round 4: _____ _____ _____ + _____ _____ _____ = _____________ ___________

Round 5: _____ _____ _____ + _____ _____ _____ = _____________ ___________

TOTAL SCORE: ___________

Multiple Bingo and Factor Bingo Instructions

Basic Activity

Multiple Bingo and Factor Bingo are versions of the traditional Bingo game. The object of the game is to mark five numbers in a row, either across, up and down, or diagonally. The numbers that can be marked are determined by drawing cards. Both versions can be played either as a whole class, with a partner, or in a small group

These games are designed to give students practice finding factors and multiples of numbers. Several variations of game play for Multiple Bingo and Factor Bingo are offered on the following page. The basic rules for both games are provided in the list below:

Multiple Bingo	Factor Bingo
Materials: • 100 (or 300) Chart (1 per player) • Multiple Bingo Cards (1 deck per playing group) • crayon or marker (1 per player) • calculators (optional)	**Materials:** • Multiplication Table (1 per player) • Factor Bingo Cards (1 deck per playing group) • crayon or marker (1 per player) • calculators (optional)
Procedure: **Step 1: Gather all the materials needed for the activity.** Each player has a 100 or 300 chart, and a crayon or marker. Each playing group has a deck of Multiple Bingo cards in a pile face down in the middle of the table. **Step 2: Draw a card from the face down pile.** Players take turns turning over a card for the group. **Step 3: Choose a number to mark.** Every player marks one number on the 100 chart that is a multiple of the number on the card drawn. Players write the original number in a corner of the square for checking later. For example, if someone turns over a 5 card, players could mark any one of the numbers 5, 10, 15, 20, 25, and so forth. In the corner of the chosen square, the player would write 5. (With the 300 chart, players will simply have more options.) * If a ***Wild Card*** is drawn, the player who turned it over can decide on the number that is to be used. The best strategy is to choose a number that helps the player's own game but doesn't help the other players. In Multiple Bingo, the most useful number to pick is often a prime number. **Step 4: Repeat the process until there is a winner.** The game continues until a player marks five numbers in a row for a Bingo. The remaining players can choose to continue until they also mark five in a row.	**Procedure:** **Step 1: Gather all the materials needed for the activity.** Each player has a Multiplication Table, and a crayon or marker. Each playing group has a deck of Factor Bingo cards in a pile face down in the middle of the table. **Step 2: Draw a card from the face down pile.** Players take turns turning over a card for the group. **Step 3: Choose a number to mark.** Every player marks one number on the Multiplication Table that is a factor of the number on the card drawn. For example, if someone turns over a 100 card, players could mark any one of the numbers 1, 2, 4, 5, 10, 20, 25, or 100. In the corner of the chosen square, the player would write 100. * If a ***Wild Card*** is drawn, the player who turned it over can decide on the number that is to be used. The best strategy is to choose a number that helps the player's own game but doesn't help the other players. In Factor Bingo, the most useful number to pick is often the exact number you want to cover. **Step 4: Repeat the process until there is a winner.** The game continues until a player marks five numbers in a row for a Bingo. The remaining players can choose to continue until they also mark five in a row.

Multiple Bingo and Factor Bingo (Cont.)

Variations of Game Play

Whole Class Game: This game can be played by the whole class. A leader, the teacher or student, draws the cards. If a Wild Card is drawn, the leader calls on a player to choose a number for the group to use. The object of the game remains the same. Play could continue until every player has covered five in a row. When the class plays as a large group, there is the option for students who are new to the game to collaborate with other students.

Limiting the Cards: For an easier version of Multiple Bingo, use only the 2, 3, 4, and 5 cards and a few wild cards. As students become comfortable with additional multiples, add more cards to the game.

For an easier version of Factor Bingo, use only the smaller Factor Bingo Cards (100, 180, 200, 60, 98, 32, 72, and 150) and a few Wild Cards.

Limiting the 100 Cards: When students first play Multiple Bingo, they will tend to use only "easy" numbers, especially the single-digit numbers and multiples of 10. Here are some ways to encourage them to use more difficult numbers:

- Block out the top row and right-hand column of the 100 chart; these numbers may not be marked.
- Establish a rule that players must start with a number near the middle of the chart. Allow students to pick one to three prime numbers greater than 20 to cover for free before the game begins.
- Give bonus points for a win that is on a diagonal. This may encourage students to notice the nines and eleven's tables on the two main diagonals.

Special Notes

Using Aids: Encourage students to use 100 charts and calculators to determine suitable factors and multiples.

Reusing the Charts: Students can use a contrasting color to play another game on the same 100 chart or Multiplication Table. Some teachers have laminated a set that can be wiped off after each game; students will need to use dry-erase markers with these laminated charts.

Related Homework Options

Playing Bingo at Home: Students can play either version of Bingo at home with copies of the appropriate materials.

Multiple Factor Bingo: 100 Chart

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Factor Bingo: Multiplication Table

	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	18	27	36	45	54	63	72	81	90	99	108
10	10	20	30	40	50	60	70	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	108	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

Multiple Bingo Cards

2	2	2	3
3	4	4	5
<u>6</u>	7	8	<u>9</u>
12	15	16	20
★ WILD CARD	★ WILD CARD	★ WILD CARD	★ WILD CARD

Multiple Bingo Cards

MULTIPLE BINGO	MULTIPLE BINGO	MULTIPLE BINGO	MULTIPLE BINGO
MULTIPLE BINGO	MULTIPLE BINGO	MULTIPLE BINGO	MULTIPLE BINGO
MULTIPLE BINGO	MULTIPLE BINGO	MULTIPLE BINGO	MULTIPLE BINGO
MULTIPLE BINGO	MULTIPLE BINGO	MULTIPLE BINGO	MULTIPLE BINGO
MULTIPLE BINGO	MULTIPLE BINGO	MULTIPLE BINGO	MULTIPLE BINGO

Factor Bingo Cards

100	180	200	60
98	32	72	150
240	144	324	225
448	396	330	450
★ WILD CARD	★ WILD CARD	★ WILD CARD	★ WILD CARD

Factor Bingo Cards

FACTOR BINGO	**FACTOR BINGO**	**FACTOR BINGO**	**FACTOR BINGO**
FACTOR BINGO	**FACTOR BINGO**	**FACTOR BINGO**	**FACTOR BINGO**
FACTOR BINGO	**FACTOR BINGO**	**FACTOR BINGO**	**FACTOR BINGO**
FACTOR BINGO	**FACTOR BINGO**	**FACTOR BINGO**	**FACTOR BINGO**
FACTOR BINGO	**FACTOR BINGO**	**FACTOR BINGO**	**FACTOR BINGO**