VIDEO VIEWING GUIDE

1. What were the issues with which the transportation planner had to contend?

2. Why was simulation so important for this particular model?

3. What were some of the assumptions that the model had to consider?

4. What was the solution that was agreed to in this model?

STUDENT MODEL: TO BUY OR LEASE

ABSTRACT

The purpose of my project is to decide whether it is cheaper to pay cash up front for a car, take out a loan on a car, or lease a car. To do this I will consider the case of a hypothetical customer who wants to buy a particular car—a 1991 Toyota MR2 Turbo Coupe with T-bar, loaded, with a sticker price of $22,000. I will assume that he has exactly $22,000 in a bank account to spend, and that he deposits $500 a month to this account. I will track his monthly payments over a five-year period and compare end results to see how he should pay for this car.

INTRODUCTION

For many people, driving a car is a necessity. Almost 90% of Americans buy the cars they drive, while the rest lease. I will compare these two options, buying and leasing, to see which is more profitable, or less costly. Many personal factors enter in to the decision of whether to buy or lease; since these factors differ from one individual to the next, I will research strictly the financial aspect of this question. Most leases and loans vary in length between two and five years. I will consider the five-year (more common) time period. I will take into account the down payment, monthly payments, and interest and insurance rates. I will consider an open-end lease, which gives the driver the right to buy the car—at a previously set price—at the end of the five-year lease. I will assume that this price is the same price at which he could sell the car if he had purchased it. I will consider a situation in which the customer has enough money to buy the car up front if he chooses not to lease or finance the car. Since I cannot consider every type of car, I have limited my research to a 1991 Toyota MR2 T-Top, a sports car with a sticker price of approximately $22,000.00. Since interest rates vary from day to day, I will work with only the rates I was given upon my first visit to the bank and car dealership.

DISCUSSION

I will decide the least expensive way of paying for a car by comparing the amount of money in the customer's bank account after 5 years with the total amount of money that he has invested in his car during the five-year period. I will assume that the customer deposits $500 a month into this specific account and that he has exactly $22,000.00, the sticker price of the MR2, in this account when he decides to get the car. The interest rate on borrowed money was quoted to me at 12%, and the interest rate on a savings account, like this one, was quoted at 8%. I will analyze the following three cases: The customer pays $22,000.00 up front in cash; the customer takes out a loan and finances the car over five years; the customer leases the car with an option to buy after five years. I will assume that a 15% down payment is required on a loan and that the down payment on a lease is the "first and last months' rent." The bank gave me a specific monthly payment for a loan, but many complex computations must be calculated when considering a lease. Banks refuse to perform these calculations unless a car is actually being bought, but the vice president of a First Union bank told me that the monthly payments on a lease are generally $100.00 less than the monthly payments on a loan. Over a 60-month (5-year) period, this is a difference of $6000.00. At the current interest rate of 12%, the monthly payments on a loan for this car are $429.19. The monthly payments on a lease for this car would therefore be $329.19. If a car is bought with a loan, life insurance must also be bought from the bank at a premium of $596.56. This insurance protects the bank in case of an accident to the purchaser, which would make him unable to pay for the car. In a leasing situation, the title of the car is held by the bank as collateral until the end of the leasing period, so this insurance is unnecessary. The Toyota dealer told me that anyone who leases a car must buy a more expensive insurance policy than anyone who purchases a car. However, both First Union Bank and U.S.A.A. insurance company said that there is no difference in the minimum limits and liability insurance that every driver must have; whether he leases or buys his car, the insurance payments need not factor into my considerations. The warranty is the same on a car whether it is bought or leased, as is the money spent on fuel, so I will consider neither of these factors. I will assume that the (bank) insurance is paid at the same time as the initial payment on the car, and that any withdrawals or deposits are made at the beginning of each month. Interest will be compounded at the end of each month.

RESULTS

Case 1: Cash up front.

Before buying the car, the customer has $22,000.00 in his bank account. Most car dealerships will give some sort of discount if the car is paid for in advance if the customer asks for one. I will assume that this customer does not ask and that he pays the dealership the sticker price of $22,000 from his account.

Let I = \$ in bank account the previous month. Then the amount of money in the current month's account is $(I + \$500) \times \left(\frac{1.08}{12}\right)$,

$500 represents the monthly deposit,

1.08 represents 8% interest compounded monthly, and

$22,000 is the total money that will be paid for the car.

Then a record of the bank account looks like **Figure 1.**

Month	$ in bank
Before purchase	$22,000.00
At purchase	$0.00

Month	$ in bank	Month	$ in bank	Month	$ in bank
1	$503.33	21	$11,305.33	41	$23,499.46
2	$1010.02	22	$11,884.03	42	$24,159.46
3	$1520.09	23	$12,466.59	43	$24,823.85
4	$2033.56	24	$13,053.04	44	$25,492.68
5	$2550.45	25	$13,643.39	45	$26,165.97
6	$3070.78	26	$14,237.68	46	$26,843.74
7	$3594.59	27	$14,835.93	47	$27,526.03
8	$4121.89	28	$15,438.17	48	$28,212.87
9	$4652.70	29	$16,044.43	49	$28,904.29
10	$5187.05	30	$16,654.72	50	$29,600.31
11	$5724.96	31	$17,269.09	51	$30,300.99
12	$6266.46	32	$17,887.55	52	$31,006.33
13	$6811.57	33	$18,510.13	53	$31,716.37
14	$7360.32	34	$19,136.87	54	$32,431.14
15	$7912.72	35	$19,767.78	55	$33,150.69
16	$8468.80	36	$20,402.90	56	$33,875.02
17	$9028.60	37	$21,042.25	57	$34,604.19
18	$9592.12	38	$21,545.59	58	$35,338.22
19	$10,159.40	39	$22,192.55	59	$36,077.14
20	$10,730.46	40	$22,843.84	60	$36,820.99

Figure 1.
The savings balance under the "cash up front" plan.

At the end of five years, the customer will have spent $22,000.00 on his car, and will have $36,820.99 in his bank account. His net gain will be the difference in these two amounts, $14,820.99. In trying to find out the monthly lease payments from the banker, the car dealer was reluctant in telling me how much the car would be worth in five years. Even though he would have to tell someone who would have an open-end lease, he would only tell me that a car is generally worth 40% of its sticker price after five years—if it's in good condition. I looked at car resale value books in the library, and this seemed to be pretty accurate. Using this information, I will assume that the MR-2 will be worth 40% of $22,000, or $8800 in 5 years.

Case 2: Takes a loan.

The customer begins with $22,000 in his bank account, but when he buys the car he must subtract $3300 for the down payment and $595.56 for the life insurance premium. I will keep two running totals for each month: his bank account, which will have a deposit of $70.91 ($500 – $429.19 monthly payment), and the amount of money he spends towards the car, the sum of which will increase by $429.19 each month. Once again, I will assume that the interest rate on the [savings] account is constant at 8% and that the interest is compounded monthly.

Let I = $ in bank the previous month. Then the amount in bank this month is $(I + \$70.91) \times \left(\frac{1.08}{12}\right)$, and a record of the bank account looks like **Figure 2.**

The customer ends up having spent $29,646.96, or $7,646.96 above the sticker price, on his car, which is worth $8800 after 5 years. In his bank account, he has $32,981.68 after five years. His net gain is equal to the money in his account minus the money he spent on the car, or $3334.72. This is $11,486.27 less than his net gain would have been if he had paid cash up front for the car.

Case 3: Lease the car.

If the customer leases the car, his monthly payments will be $329.19, and he will add $170.91 ($500 deposit – monthly payment) to his bank account each month. Rather than having a certain percentage required as a down payment, he will have to put down twice his monthly payment, or $658.38. I will assume that the [savings] interest rate is still 8% compounded monthly and track these two running totals over the course of the 60-month period. Every lease has certain requirements about the condition the car must be in upon its return, and additional money must be paid to the dealer if the car is not in good enough shape, or if too many miles have been driven. I will assume that the driver will meet all of these specifications, and will buy the car from the dealership at $8800 after the five years. This is the same price that he would be able to sell the five-year old car at if he owned it. Once he owns the car, his situation can be accurately compared to the first two situations.

Month	$ in bank	$ toward car
Before purchase	$22,000.00	0
At purchase	$18,104.44	$3895.56

Month	$ in bank	$ toward car	Month	$ in bank	$ toward car
1	$18,296.52	$4324.75	31	$24,694.56	$17,200.45
2	$18,489.89	$4753.94	32	$25,564.78	$17,629.64
3	$18,684.53	$5183.13	33	$25,806.59	$18,058.83
4	$18,880.47	$5612.32	34	$26,050.02	$18,488.02
5	$19,077.73	$6041.51	35	$26,295.07	$18,917.21
6	$19,276.29	$6470.70	36	$26,541.75	$19,346.40
7	$19,476.18	$6899.89	37	$26,790.07	$19,775.59
8	$19,677.41	$7329.08	38	$27,040.06	$20,204.78
9	$19,879.97	$7758.27	39	$27,291.71	$20,633.97
10	$20,083.89	$8187.46	40	$27,545.04	$21,063.16
11	$20,289.16	$8616.65	41	$27,800.06	$21,492.35
12	$20,495.81	$9045.84	42	$28,056.77	$21,921.54
13	$20,703.83	$9475.03	43	$28,315.20	$22,350.73
14	$20,913.24	$9904.22	44	$28,575.35	$22,779.92
15	$21,124.04	$10,333.41	45	$28,837.24	$23,209.11
16	$21,336.25	$10,762.60	46	$29,100.87	$23,638.30
17	$21,549.88	$11,191.79	47	$29,366.26	$24,067.49
18	$21,620.79	$11,620.98	48	$29,633.41	$24,496.68
19	$21,981.41	$12,050.17	49	$29,902.35	$24,925.87
20	$22,199.33	$12,479.36	50	$30,173.08	$25,355.06
21	$22,418.71	$12,908.55	51	$30,445.62	$25,784.25
22	$22,639.55	$13,337.74	52	$30,719.97	$26,213.44
23	$22,861.87	$13,766.93	53	$30,996.16	$26,642.63
24	$23,085.66	$14,196.12	54	$31,274.18	$27,071.82
25	$23,310.95	$14,625.31	55	$31,554.06	$27,501.01
26	$23,537.74	$15,054.50	56	$31,835.80	$27,930.20
27	$23,766.04	$15,483.69	57	$32,119.42	$28,359.39
28	$23,995.86	$15,912.88	58	$32,404.93	$28,788.58
29	$24,227.21	$16,342.07	59	$32,692.35	$29,217.77
30	$24,460.11	$16,771.26	60	$32,981.68	$29,646.96

Figure 2.
The savings balance under the "takes a loan" plan.

Let I = $ in bank the previous month. Then the amount in bank this month is $(I + 170.91) \times \left(\frac{1.08}{12}\right)$, and a [partial] record of the bank account looks like **Figure 3.**

Month	$ in bank	$ toward car
Before purchase	$22,000.00	0
At purchase	$21,341.62	$658.38

Month	$ in bank	$ toward car	Month	$ in bank	$ toward car
1	$21,655.95	$987.57	31	$32,126.00	$10,863.27
2	$21,972.37	$1316.76	32	$32,512.23	$11,192.46
3	$22,290.90	$1645.95	33	$32,901.02	$11,521.65
4	$22,611.56	$1975.14	34	$33,292.41	$11,850.84
5	$22,934.35	$2304.33	35	$33,686.41	$12,180.03
6	$23,259.29	$2633.52	36	$34,083.04	$12,509.22
7	$23,586.41	$2962.71	37	$34,482.31	$12,838.41
8	$23,915.70	$3291.90	38	$34,884.24	$13,167.60
9	$24,247.19	$3621.09	39	$35,288.85	$13,496.79
10	$24,580.88	$3950.28	40	$35,696.16	$13,825.98
11	$24,916.80	$4279.47	41	$36,106.18	$14,155.17
12	$25,254.97	$4608.66	42	$36,518.94	$14,484.36
13	$25,595.38	$4937.85	43	$36,934.45	$14,813.55
14	$25,938.07	$5267.04	44	$37,352.73	$15,142.74
15	$26,283.04	$5596.23	45	$37,773.80	$15,471.93
16	$26,630.31	$5925.42	46	$38,197.67	$15,801.12
17	$26,979.89	$6254.61	47	$38,624.37	$16,130.31
18	$27,331.82	$6583.80	48	$39,053.92	$16,459.50
19	$27,686.07	$6912.99	49	$39,486.32	$16,788.69
20	$28,042.69	$7242.18	50	$39,921.62	$17,117.88
21	$28,401.70	$7571.37	51	$40,359.81	$17,447.07
22	$28,763.09	$7900.56	52	$40,800.92	$17,776.26
23	$29,126.89	$8229.75	53	$41,244.98	$18,105.45
24	$29,493.12	$8558.94	54	$41,692.00	$18,434.64
25	$29,861.79	$8888.13	55	$42,142.00	$18,763.83
26	$30,232.92	$9217.32	56	$42,594.99	$19,093.02
27	$30,606.52	$9546.51	57	$43,051.00	$19,422.21
28	$30,982.61	$9875.70	58	$43,510.06	$19,751.40
29	$31,361.22	$10,204.89	59	$43,972.18	$20,080.59
30	$31,742.34	$10,534.08	60	$44,437.37	$20,409.78

Figure 3.
The savings balance under the "lease a car" plan.

At the end of this five-year period, if the customer decides not to buy the car, he will have put $20,409.78 towards it and will have $44,437.37 in his bank account. This will give him a net gain of $24,027.67. However, if I am to compare his situation to that of buying the car, then I will assume that he buys the car from the dealership at the set price of $8800 at the end of the five years. Now, his net gain would be $15,227.59

After 5 years:

	$ in bank	*$ in car*	*Net gain*
Cash up front	$36,820.99	$22,000.00	$14,820.99
Buy with loan	$32,981.68	$29,646.96	$3,334.72
Lease the car	$44,437.37	$29,209.78	$15,227.59

CONCLUSIONS AND RECOMMENDATIONS

Obviously, paying off a loan over a five-year period is the most expensive way to pay for a car. My calculations have shown that the best way to pay for a car is cash up front. My hypothetical buyer would have over three times as much money after five years if he paid up front rather than took out a loan for the MR2; he would have twice as much money than if he had leased the car. These numbers depend directly on the accuracy of my monthly payment estimates and on the interest rate remaining 8% throughout the five years. If the interest rate were less than 8% then paying cash up front would be even more profitable; his money in the bank is not as heavily relied on for profits from interest as are the other two options. It appears that leasing a car is the better alternative to taking out a loan, but if the difference in monthly payments of the two is substantially less than $100 a month, then the two options would be more equivalent if the dealer had offered the customer a discount for paying up front then it would be even more profitable. The exact amount of money that the MR2 will be worth in five years could change the outcome of the problem, possibly making buying the car a better choice than leasing it. If the car is expected to be worth $11,000 or more in five years, then leasing would become the least profitable option. Also, if you own a car you are not penalized for putting too many miles on it or for having it not meet certain standards of cleanliness. If these expectations are not met, there is a surcharge on the lease (between 8 and 12 cents for every extra mile + $ for poor condition) that must be paid at the end of the five-year period. Other than finance, the most important question here is "does the customer want to continue driving the same car after the five years are over?" If the answer is no, then he should definitely lease the car to save the hassle and cost of trying to sell a used car. Realistically, most people are unable to pay up front and in full at the time of purchase. My calculations considered the same car in three different situations. Many of the people who lease are actually driving more expensive cars than they could afford to buy. The results of my study were shocking since the overwhelming majority of car drivers take out loans to buy their car. I was especially surprised to see that leasing saved money over taking out a loan to pay for the car since only about 10% of the drivers lease their automobiles. This might stem from the fact that not all leases are open ended and the drivers might have a strong desire to own the cars they drive. Or they might be positive that they could not meet the mileage restrictions on a leased car. The car dealerships do not seem to promote leasing, although they do offer it as an alternative to buying.

EXTENSIONS

This model had to involve very specific, unchanging values in order for me to make my calculations. I considered only a savings account rather than a money market account, stock investment, or government bond. I did not "shop around" for a savings account with greater than 8% interest simply because the banker told me that interest rates often depend on a customer's individual relationship with the bank, and I used the relationship that my family had with First Union. I was surprised to find out that leasing was more profitable than taking out a loan. I know I shortchanged the customer in the cash-up-front situation, because I started his account at $0 after he paid for the car. I did this for the sake of keeping my numbers small, but I realize that he really lost out when it came to the interest his money was earning. The instability in my project lies in the estimates that I had simply to take on faith from the bank and car dealership; this car's value will probably not be exactly 40% of $20,000 in five years, and the monthly payments on a lease of this car are probably slightly more or less than $329.19. I had hoped to discover that paying cash up front was the most economical solution because that is how my family has always bought its cars. I wish I had had the opportunity (time) to experiment with the price of the car, amount of money in the bank, and interest rate of the account. The only question that I am left with is, "Why aren't more people leasing their cars rather than taking out loans?"

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SAMPLE MODEL: MINIMIZING ELEVATOR TIME

THE PROBLEM:

Poor elevator service in an office building. The manager is unwilling to construct additional elevators, being convinced that the problem can be resolved through better scheduling.

FACTS:

- There are five working floors and an unused ground floor.
- 80 people work on each floor.
- There are four elevators in the building.
- Each elevator can hold 10 people.
- Elevators take 22 seconds to load or unload.
- Elevators take 3 seconds to travel between floors.

ASSUMPTIONS:

- Each elevator will carry its capacity (10 people) each trip.
- No elevator will pick up or deliver on its way down.
- All people arrive at the office building at about the same time and will continuously fill the elevators.

PURPOSE:

- To minimize the amount of time it takes the four elevators to deliver the 400 people to their respective floors.

SOLUTION:

- If each elevator stops at each floor, four elevators can deliver 400 people if each elevator carries its capacity 10 times.
- Every stop takes 22 seconds. An elevator must stop 6 times (including initial loading) when it travels to all floors.
- Travel time between successive floors is 3 seconds. Thus, a two-way trip takes 30 seconds of travel time, plus its stopping time. Therefore, 10 round-trips for each elevator would take it

  $((22 \times 6) + (30)) \times 10 = 1620$ sec., or 27 minutes.

Fewer stops would decrease this time. To make the fewest stops, each elevator would make only two stops per trip. Since there are not enough elevators to do this, a different schedule should be implemented.

Require all four elevators to make two trips to the first floor, thus delivering the 80 people in two loads of forty people each. Repeat this process for each floor.

Each trip takes 22 seconds to load and 22 seconds to unload, plus travel time up and down at 3 seconds per floor each way. Therefore the time taken to deliver two round trips to floor X is expressed by the formula $2(44 + 6X)$, or $88 + 12X$. So total time looks like:

Floor 1	88 + 12
Floor 2	88 + 24
Floor 3	88 + 36
Floor 4	88 + 48
Floor 5	88 + 60
Total	440 + 180 = 620 sec. or 10 min 20 sec.

While this more than halves the amount of time, this method is very impractical. A schedule would be needed, and a person missing his elevator would not be able to catch one going to his floor until after morning rush hour. This method would also require an elevator attendant to ensure smooth operation of the elevator scheduling.

A more practical solution can be realized without confusion or scheduling. Perhaps the fastest way to transport the people to their destinations would be to dedicate elevators for each floor. However, since elevators cannot be added, at least one elevator would have to carry people to two floors. Different solutions were tried.

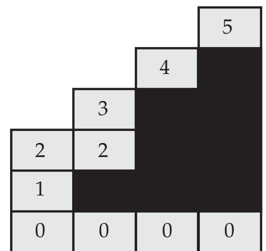
For stops at adjacent floors, where X is the floor of the first (lower) stop, the following times apply:

Initial loading	22 seconds
Trip to floor	$3x$
Unloading at first stop	22
Trip to next floor	3
Unloading at second stop	22
Trip back	$3(x + 1)$
Equation for time	$6x + 72$
Total time for one round trip	$6x + 72$

With floors 5, 4, and 3 having their own elevators, and floors 1 and 2 sharing an elevator, the minimum amount of time necessary to transport all the people to their respective destinations was found to be 1248 sec. or 20.8 minutes. This is the lowest time for the sharing of one elevator by two floors.

FINAL SOLUTION:

A better solution is to have two elevators (instead of only one) take the load of the extra floor. After trying other combinations, it was determined that elevator 1 should serve floors 1 and 2; elevator 2 should serve floors 2 and 3. Floors 4 and 5 would have their own elevators.



Elevator	# 1	# 2	# 3	# 4
Time to complete load	936 sec.	1008 sec.	544 sec.	592 sec.

■ Floors not served

Recall that the time taken for one round trip to floor X is given by $44 + 6X$. Therefore, elevator 4 takes 592 seconds to carry the 80 people to floor 5:

$44 + 6(5) = 74$ seconds per round trip.

74×8 trips = 592 seconds.

Likewise, elevator 3 takes 544 seconds to carry the 80 people to floor 4:

$44 + 6(4) = 68$ seconds per round trip, for $68 \times 8 = 544$ seconds.

Assume that elevator 2 carries $\frac{1}{2}$ of the people from floor 2 and all of the people from floor 3. Recall that the time taken to deliver on adjacent floors (X and $X + 1$) is given by $6X + 72$.

Thus, elevator 2 takes 1008 seconds to carry the 120 people to floors 2 and 3:

$6(2) + 72 = 84$ per round trip, and $84 \times 12 = 1008$ seconds. (There are 12 trips—80 people to 3, 40 to 2.)

In the same fashion, elevator 1 carries $\frac{1}{2}$ of the people from floor 2 and all of the people to floor 1. Therefore, elevator 1 takes 936 seconds to carry the 120 people to floors 1 and 2:

$6(1) + 72 = 78$; $78 \times 12 = 936$ seconds.

The total time for transporting all the people to their offices using this scheme is 1008 seconds (16.8 minutes), since this is the largest amount of time any given elevator takes to deliver its full load.

IN CONCLUSION:

The final solution shaves nearly 10 minutes off of the original scenario. The first solution was even better but entails too many constraints and could cause major confusion. The second solution is much more practical and fool-proof but assumes that one half of the people going to the second floor use elevator 1 and the other half use elevator 2. This assumption is reasonable since passengers headed for floor 2 will always fill an empty elevator. Even if elevator 2 must carry all 160 people to the second and third floors, this solution is faster than the original scenario by about 5 minutes.

Further Note: If elevator 3 converts, at the end of the first 8 round-trips, to taking the same route as elevator 2 and elevator 4 converts to the same route as elevator 1, it is possible to cut this time down to 826 seconds or 13 minutes 46 seconds. However, this may not be a practical solution, since people may get confused as to where the elevators are going.

SAMPLE MODEL: PASTURELAND

THE PROBLEM

A rancher has a prize bull and some cows on his ranch. He has a large area for pasture that includes a stream running along one edge. He must divide the pasture into two regions, one region large enough for the cows and a smaller region to hold the bull. The cows' grazing pasture must be at least 10,000 sq. feet, and the bull's grazing pasture must be at least 1000 sq. feet. The shape of the pasture is basically a rectangle measuring 120 feet by 150 feet. The stream runs all the way along the 120-foot side. Fencing costs $5/foot and each fence post costs $10. Any straight edge of fence requires a post every 20 feet, and any curved length of fence requires a post every 10 feet. Help this rancher minimize the total cost of fencing.

CONSIDERATIONS AND ASSUMPTIONS

I first looked at the overall area of the pasture to see if there was enough land available for both the bull and cows. Assuming a perfect rectangle, the area of 18,000 square feet in the pasture will hold both.

I next considered the stream. If this stream is not traversible by the cattle, I could use the bank of the stream as part of the fence. However, since the term stream was used instead of the term river, I made the assumption that this was not the case. I therefore assume that I have to fence all sides of the pastures both for the bull and for the cows.

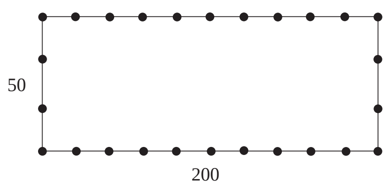
I next considered combining some of the fencing—that is, enclosing one within the other or using a common side. This would save some of the fencing and some of the posts. As I considered this, I realized that cows and bulls need to be separated by more than just a fence. They need some land between them as well. If not, the bull would tear the fence down to get to the cows. If this were not an issue, it would be possible to include all the cattle in a fenced area of 11,000 square feet and not worry about separate pens.

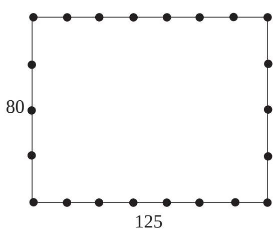
To summarize my assumptions:

1. I am assuming that all sides of each of the pens will need fencing.
2. I am assuming that there will be no common fence. That is, there will be two separate pens.
3. I am assuming that I will enclose the smallest amounts of pasture necessary both for the cows and for the bull (≥1000 for the bull and ≥10000 for the cows) consistent with assumption #4.

4. I will round all side measurements up to the next foot. This is to make sure I have enough area enclosed in the pens and to make measurements simpler.

Given these assumptions, I began to look at different shapes that might be used for the two pens. I wanted to see if there were some shapes that were cheaper than others to build. I decided to limit my examination to 3 different types of figures—a rectangle, a triangle, and a circle. I hoped that by examining these figures one would emerge as a distinct winner in reducing the cost of building the pens.





RECTANGLE

I first looked at a rectangle, assuming an area of 10,000 square feet. I set up a spreadsheet that would examine different dimensions to see if one was cheaper to build. I varied one side in increments of 10 feet to see if a pattern emerged. It was only necessary to go from 10 to 100, since the measurements of the sides would just begin reversing after this. The number of posts needed along one side of length *L*, including *one* corner, is given by $\lceil L/20 \rceil$. The formula that I used on the spreadsheet to calculate the value in row 5 of column 5 (column 6 was similar) was –Int(–B5/20). The fencing cost is calculated by multiplying $5 by the perimeter, and the post cost is found by multiplying $10 times the number of posts.

Cow pen 10,000							
Rectangle area	Short side	Long side	Perimeter	Posts	Fencing costs	Post cost	Total cost
10,000	10	1000	2020	102	$10,100.00	$1020.00	$11,120.00
10,000	20	500	1040	52	$5200.00	$520.00	$5720.00
10,020	30	334	728	38	$3640.00	$380.00	$4020.00
10,000	40	250	580	30	$2900.00	$300.00	$3200.00
10,000	50	200	500	26	$2500.00	$260.00	$2760.00
10,020	60	167	454	24	$2270.00	$240.00	$2510.00
10,010	70	143	426	24	$2130.00	$240.00	$2370.00
10,000	80	125	410	22	$2050.00	$220.00	$2270.00
10,080	90	112	404	22	$2020.00	$220.00	$2240.00
10,000	100	100	400	20	$2000.00	$200.00	$2200.00

The cost seems lowest when the figure created is a square. I next checked the bull pen for the rectangular configuration.

Bull pen 1000							
Rectangle area	**Short side**	**Long side**	**Perimeter**	**Posts**	**Fencing costs**	**Post cost**	**Total cost**
1000	5	200	410	12	$2050.00	$120.00	$1220.00
1000	10	100	220	10	$1100.00	$100.00	$920.00
1005	15	67	164	8	$820.00	$80.00	$780.00
1000	20	50	140	8	$700.00	$80.00	$730.00
1000	25	40	130	8	$650.00	$80.00	$720.00
1020	30	34	128	8	$640.00	$80.00	$720.00
1023	31	33	128	8	$640.00	$80.00	$720.00
1024	32	32	128	8	$640.00	$80.00	$720.00
1023	33	31	128	8	$640.00	$80.00	$720.00
1020	34	30	128	8	$640.00	$80.00	$720.00
1015	35	29	128	8	$640.00	$80.00	$720.00
1008	36	28	128	8	$640.00	$80.00	$720.00
1036	37	28	130	8	$650.00	$80.00	$720.00
1026	38	27	130	8	$650.00	$80.00	$730.00

Note that a number of different dimensions lead to exactly the same costs for the bull pen. Therefore, I selected the one that uses the most area for the lowest cost, since I want to have enough pasture to feed the bull.

I needed to check to see if these two figures (a 100 x 100 pen and a 32 x 32 pen) would fit into the same 120 x 150 pastureland and still fit the assumptions that I made. When I looked at this, I was able to place both in the pastureland and have nothing in common if that were desired. It would be possible to reduce the cost by one post by having the pens share a corner. However, since there was no clear reason for doing this, I decided to keep the pens separate.

My conclusion, therefore, is that if I use two rectangles, the estimated cost for creating the two pens would be $2200 + $720 = $2920.

CIRCLE

I next looked at producing a pair of circular pens.

Circle	Total area	Radius	Circumference	# of posts	Fencing costs	Post cost	Total cost
Cows	10,000	57	358	36	$1790.00	$360.00	$2150.00
Area used		10,207					
Bull	1000	18	113	12	$565.00	$120.00	$685.00
Area used		1017				**Total of both**	$2835.00

These values are based on rounding the radius to the next highest integer value to guarantee I have at least 10,000 and 1000 square feet in the pens. The next question I asked was: Is it possible to create these two pens in the same pastureland? A circle of radius 57 feet can be enclosed in a square of 114 feet on a side. A circle of radius 18 feet can be enclosed in a square with 36-foot sides. Therefore, both should fit within a rectangle with measurements of 120 by 150. Depending on how the radii are located, there may be one location where the two circles touch. However, since I rounded up for the radii, I should be able to shorten one, or both, slightly so that this is not a problem.

My conclusion is that if I use two circles, the estimated cost for creating the two pens would be $2150 + $685 = $2835.

TRIANGLE

For this part, since the square gave me the best answer on the rectangle part, I will assume that an equilateral triangle will be the best answer when calculating triangle results.

Triangle	Side	Perimeter	Height	Actual area	Posts	Total
Equilateral total area 10,000						
	152	456	132	10,004		$2280.00
Posts	8	24			24	$240.00
					Cow	$2520.00
Equilateral total area 1000						
	49	147	43	1039		$735.00
Posts	3	9			9	$90.00
					Bull	$825.0
					Total	$3345.00

The large triangle can be enclosed in rectangle of 152 x 132 feet. This becomes a problem since the shorter side of the pasture is only 120 feet long. There may be a possibility of fitting other types of triangles into this region. However, if my assumption is correct—that the least expensive of all triangles is equilateral and the equilateral triangle costs $3345—then there is no sense in checking other types since this is already more than the cost of a rectangular or circular configuration.

CONCLUSION

Based on the assumptions made and the calculations performed, the least expensive way to create these two pens is to use two circles, one with a radius of approximately 57 feet and the other with a radius of approximately 18 feet. The cost for this would be around $2835.00. It seems clear that if you are using two separate pens, that the least expensive way to construct them is to use circles, given the three choices I selected.

VARIATIONS

If the cattle cannot cross the stream, it would be possible to use the stream as one of the fences. This would reduce the overall cost and would require recalculating the totals.

If the bull would not break through a shared fence, it would be possible to reduce the costs by using one or more common sides. This would allow a reduction of costs and would allow you to examine a combination of figures.

Although this analysis used separate spreadsheet calculations, it is possible to create functions that describe the different costs based on the area of the overall pasture acreage and side or radius. Graphing these cost equations could help approximate the cheapest cost for producing the two pens. Finding such functions would allow you to use this process for differing initial scenarios, and continuous variations in sides and radii.

SAMPLE MODEL: FACULTY

THE PROBLEM

When confronted with a rise of 142 students in a school of 480, and a capacity for 7 new teachers, in what departments should the new teachers be placed? Placing the new teachers should maintain the ideal student-to-teacher ratio. The current makeup of the (student:teacher) enrollment in each department is: Art (99:1); Biology (319:4); Chemistry (294:3); English (480:5); French (122:1); German (51:1); Spanish (110:1); Mathematics (613:6); Music (95:1); Physics (291:3); and Social Studies (363:4).

INVESTIGATIONS INTO THE SCHEDULING PROBLEM

The initial assumptions that we made for this problem were:

1. No departments will have a larger student:teacher ratio than 125:1.
2. The new students will be distributed along the same ratios that currently exist. That is, for example, approximately the same percentage of the new students will take mathematics as the current percentage.
3. Each new enrollee will take 6 classes.
4. It is clear that this school allows students to take more than one class in a department (math 613) so we will allow the new students this same option.
5. When there is a fractional student increase we will determine whether it makes sense to add an additional student.

The existing population (student and teacher) is listed in the table below.

Subjects	Current enrollment	Number of teachers	Student/teacher ratio	Overall % enrollment/total classes
Art	99	1	99.000	3.49%
Biology	319	4	79.750	11.24%
Chemistry	294	3	98.000	10.36%
English	480	5	96.000	16.92%
French	122	1	122.000	4.30%
German	51	1	51.000	1.80%
Spanish	110	1	110.000	3.88%
Math	613	6	102.167	21.61%
Music	95	1	95.000	3.35%
Physics	291	3	97.000	10.26%
Social Studies	363	4	90.750	12.80%
Totals	2837	30	94.567	

We looked at where these new students will go in relation to the existing population. Using the current overall percentages for each department (3.49% for art, times the 852 new student hours, for example), we arrived at the following set-up for each department. (New hours were rounded to the nearest integer.)

Subjects	Current enrollment	Percent enrollment	New student hours	New student hours (rounded)	Total student hours
Art	99	3.49%	29.735	30	129
Biology	319	11.24%	95.765	96	415
Chemistry	294	10.36%	88.267	88	382
English	480	16.92%	144.158	144	624
French	122	4.30%	36.636	37	159
German	51	1.80%	15.336	15	66
Spanish	110	3.88%	33.058	33	143
Math	613	21.61%	184.117	184	797
Music	95	3.35%	28.542	29	124
Physics	291	10.26%	87.415	87	378
Social Studies	363	12.80%	109.056	109	472
Totals	2837	100%	852.085	852	

We now need to determine in which departments additional teachers are needed, and how many. To do this, we first looked at what would happen if we didn't increase the total number of teachers.

Subjects	Total student hours	Number of teachers	Student hour per teacher
Art	129	1	129
Biology	415	4	103.75
Chemistry	382	3	127.3333333
English	624	5	124.8
French	159	1	159
German	66	1	66
Spanish	143	1	143
Math	797	6	132.8333333
Music	124	1	124
Physics	378	3	126
Social Studies	472	4	118
Totals	3689	30	122.9666667

We then set up a spreadsheet and began to add teachers in various ways. The things that we were looking at were fairness and balance. Balance was a bit easier to observe since we just had to look to see if the ratios were close to each other. Fairness was a bit more difficult since we knew we could never achieve perfect balance and have all departments equally happy. We started by looking at those departments with student hours per teacher higher than the school-wide average (Art, Chemistry, English, French, Spanish, Math, Music, and Physics). Since the alloment of new teachers is too small to permit adding one to each of these departments, we decided to add one teacher to each of these departments that already had more than one teacher.

Subjects	Total student hours	Number of teachers	Student hours per teacher	Add	Student hours per teacher
Art	129	1	129		129
Biology	415	4	103.75		103.75
Chemistry	382	3	127.3333333	1	95.5
English	624	5	124.8	1	104
French	159	1	159		159
German	66	1	66		66
Spanish	143	1	143		143
Math	797	6	132.8333333	1	113.8571429
Music	124	1	124		124
Physics	378	3	126	1	94.5
Social Studies	472	4	118		118
Totals	3689	30		4	

We had 3 more teachers to add. Examining the new Hours per teacher ratios, we decided to begin with the highest and assign the remaining teachers. Therefore French, Spanish, and Art will get the last 3 teachers. The results are as follows:

Subjects	Total student hours	Number of teachers	Student hours per teacher	Add	Student hours per teacher
Art	129	1	129.000	1	64.500
Biology	415	4	103.750		103.750
Chemistry	382	3	127.333	1	95.500
English	624	5	124.800	1	104.000
French	159	1	159.000	1	79.500
German	66	1	66.000		66.000
Spanish	143	1	143.000	1	71.500
Math	797	6	132.833	1	113.857
Music	124	1	124.000		124.000
Physics	378	3	126.000	1	94.500
Social Studies	472	4	118.000		118.000
Totals	3689	30		7	

Obviously, there will be some departments not satisfied with the results. Social Studies, for instance, increases enrollment by over 100 student hours and gets no additional teacher.

CONCLUSIONS

Using the assumptions that we listed at the beginning, the 7 teachers hired should be in Art, Chemistry, English, French, Spanish, Math, and Physics. We did wonder why Physical Education wasn't listed among the departments.

VARIATIONS

If it were possible to hire a teacher who could teach in more than one department, it would open up many more options. It then might be necessary to look at individual courses. If Physical Education or some other department were added, it would also be necessary to staff it.

SAMPLE MODEL: FLU EPIDEMIC

THE PROBLEM

A certain population of 20,000 people is hit by a strain of flu. One percent of the susceptible population is stricken each day. The flu effects last 5 days, after which the person is then immune to this virus. Ten percent of the population is naturally immune to the virus.

a) If the epidemic were allowed to run its course, how long would it take for the entire population to become immune?

b) The Health Service would like to institute an immunization program in an attempt to knock out the epidemic quickly—within a month, if possible. However, they would also like to inoculate as few people as possible because the serum does occasionally make people feel sick. Devise an immunization procedure that will meet these guidelines.

ASSUMPTIONS

- There are 30 days in a month.
- Once infected, a person is no longer part of the susceptible population.
- Getting sick from the inoculation is no different than getting sick from having the disease.
- Immunity from inoculation is effective immediately.
- No inoculations will be given to people who already have the disease.
- Inoculations occur at the start of each day, before any new infections for that day.
- There is no limit to the supply of vaccine or of staff to administer it.
- Calculations will be done without intermediate rounding.

SOLUTION

a) If the epidemic were allowed to run its course, how long would it take for the entire population to become immune?

10% immune => 0.10 x 20,000 = 2000 people are immune initially

=> 20,000 – 2000 = 18,000 are susceptible

If 1% of the susceptible population is infected each day, then 99% of the susceptible population is still susceptible at the end of the day.

$18{,}000(0.99)^n$ people are still susceptible after n days (assuming no intermediate rounding).

We decided to find the day on which 18,000 $(0.99)^n < 1$. Using a graphing calculator, we entered 18000 $(0.99)^x$ as one function and 1 as another function. We used the window $x$min = 500, $x$max = 2000, $y$min = 0, $y$max = 118 (since 18,000 $(0.99)^{500}$ is around 118) and calculated the intersection of these two functions. We found it to be around $x = 974.9$ and $y = 1$. Logs confirmed that result. This means that after around 975 days, the entire population has either gotten the flu or is immune. Five days after that 980 days, the flu should have run its course.

[Using 18,000 $(0.99)^n < 0.5$, instead, gives a duration of about 1049 days. In light of this wide range of reasonable values, together with the uncertainty of how rounding and randomness affect real results, perhaps the best answer here is "about 1000 days." That's almost 3 years!]

b) The Health Service would like to institute an immunization program in an attempt to knock out the epidemic quickly, within a month, if possible. However, they would also like to inoculate as few people as possible because the serum does occasionally make people feel sick.

Initially, 2000 people are immune, 18,000 people are susceptible.

$18{,}000(0.99)^n$ people are still susceptible after n days => in 30 days, the number of susceptible people will be $18{,}000(0.99)^{30} = 13{,}315$.

If the flu runs unchallenged, there will be 13,315 people to inoculate on the 30th day.

If the inoculations were evenly spaced over the month, what number of inoculations would result in 0 people being susceptible at the end of day 30?

Using a spreadsheet and trial and error, we found that by inoculating 511 people on days 1 through 29 and by inoculating 530 people on day 30, all would have either contracted the flu or be immune to it. The spreadsheet calculates the values in the following way:

- Immune—Sum of the previous immune, the number inoculated, and the number that contracted the flu.
- Number Inoculated—this was the explanatory variable. We changed this value to try to get to 0 contracting after 30 days.
- Number Contracting—1% of the previous day's Number Susceptible.
- Number Susceptible—18,000 minus the sum of the Immune, Inoculated, and Contracting.

Day	Number of immune	Number of inoculated	Number contracting	Number susceptible
0	2000	0	0	18,000
1	2000	511	180	17,309
2	2691	511	173	16,625
3	3375	511	166	15,948
4	4052	511	159	15,277
5	4723	511	153	14,613
6	5387	511	146	13,956
7	6044	511	140	13,306
8	6694	511	133	12,662
9	7338	511	127	12,024
10	7976	511	120	11,393
11	8607	511	114	10,768
12	9232	511	108	10,149
13	9851	511	101	9537
14	10,463	511	95	8930
15	11,070	511	89	8330
16	11,670	511	83	7736
17	12,264	511	77	7147
18	12,853	511	71	6565
19	13,435	511	66	5988
20	14,012	511	60	5417
21	14,583	511	54	4852
22	15,148	511	49	4293
23	15,707	511	43	3739
24	16,261	511	37	3190
25	16,810	511	32	2647
26	17,353	511	26	2110
27	17,890	511	21	1578
28	18,422	511	16	1051
29	18,949	511	11	530
30	19,470	530	0	0
		15,349	2651	

CONCLUSIONS

The model that we created indicates that we need to inoculate a total of 15,349 over 30 days, and that 2651 people will contract the flu during that time.

VARIATIONS

Since we know that our model predicts 2651 people contracting the flu and 15,349 getting the vaccine, we could try to inoculate a larger group to start with and then reduce this number further into the month. This would be interesting for further investigation.