

CHAPTER

3



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Many real-world problems involve objects that are connected in some way. For example, cities are connected by airline flights. People are connected because they are relatives, but also because they work at the same place or live in the same neighborhood. Many modeling problems arise in these situations: a person who is traveling to several cities wants to use the flights that have the lowest total cost; the phone company wants to connect homes and businesses in the most efficient way; a company wants to schedule its employees in a way that makes the best use of their skills and company facilities.

Often in your study of mathematics you have seen that the way a problem is represented can be important to the solution of the problem. For example, when you analyze a relationship that you have described with a mathematical function, a graph of the function helps you see the relationship and answer questions about it.

It is not always possible to describe a relationship with a function. Modelers sometimes use another type of graph to show relationships among objects. In this chapter, you apply this new type of graph to real-world problems and develop problem-solving procedures called algorithms.



A Management Problem

LESSON ONE

Connections

Key Concepts

Graph

Network

Vertex

Edge

Compatibility graph

Conflict graph

Optimization problems

Digraph

Weighted graph

Imagine that you are the manager of a small company. The company rents equipment that it does not use often. As part of your job, you must arrange to rent this equipment as cheaply as possible.

Several current contracts require that you rent time on a large computer. The computer can handle certain kinds of tasks at the same time, but not others.

Figure 3.1 shows which contracts can share computer time. Each contract requires an hour, whether or not it shares time with another contract. Of course, you could rent the computer for five hours, but that may not be necessary. If you are doing your job well, you will assign more than one task at a time to the computer and rent it for fewer than five hours.

Contract	Can share time with
A	B, C, E
B	A, C, D
C	A, B, D
D	B, C
E	A

Figure 3.1.
A table of five contracts.

This situation is typical of many that occur often in businesses and other organizations. Precious resources such as computer time must be managed well. If resources are poorly managed, the business and its employees suffer. If they are managed well, both prosper. In this lesson, you consider a visual way to represent situations like this one.

Activity 3.1: A Graph Model

In this activity, you learn to use graphs to represent situations.

The situation in the preparation reading can be represented with a mathematical model called a **graph** (sometimes called a network). This type of graph is made up of points called **vertices** (the plural of **vertex**) and connecting lines called **edges**. (The connecting lines do not have to be straight.) When these graphs are used to model real-world situations, the vertices usually represent objects and the edges usually represent relationships or connections among the objects.

TAKE NOTE

Do not confuse these graphs with the function graphs that a graphing calculator produces. Also, do not confuse them with statistical graphs such as bar graphs, scatter plots, or pie charts.

One way to make a graph to represent the situation in the preparation reading is to first draw a vertex for each contract, as shown in **Figure 3.2**. To keep track of which vertex represents which contract, label each vertex with a contract's name.

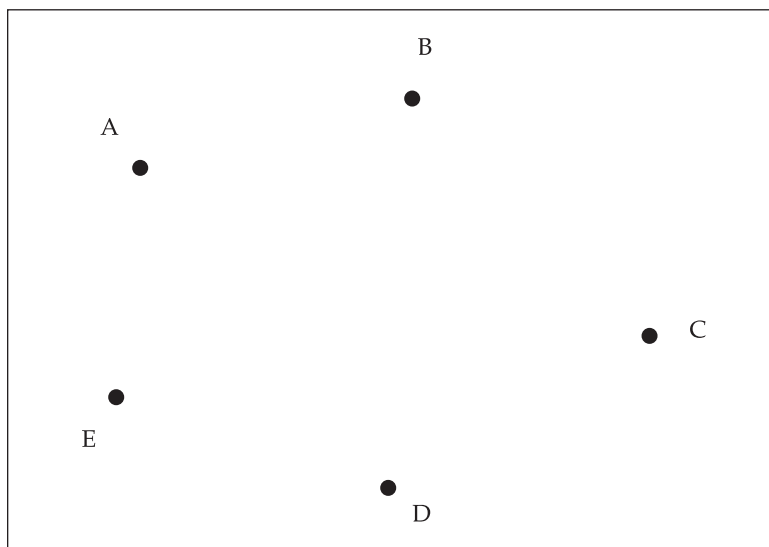


Figure 3.2.
The first step in drawing a graph.

The next step is to draw edges between vertices according to some criterion. For example, you could draw an edge if two contracts can share computer time. But you could choose a different criterion. You could draw an edge if two contracts cannot share computer time.

A graph in which edges are drawn when the vertices are compatible is called a **compatibility graph**. A graph in which edges are drawn when the vertices are not compatible is called a **conflict graph**.

MODELING NOTE

There isn't a single correct way to draw a graph for a situation. A good graph is one that clearly shows the information needed to solve some problem. Skill at constructing good graphs comes with practice.

TAKE NOTE

You can draw edges that cross each other because sometimes it is not possible to avoid doing so. However, if you draw an edge across another edge, the point at which they cross is not a vertex. You can curve edges to avoid crossings when possible.

MODELING NOTE

A graph often is a useful tool in situations that are similar to this one, when someone is trying to find the best (optimal) way to do something. Problems concerned with the best way to do something are called optimization problems. For example, the manager in this activity is trying to find an optimal way to schedule computer time.

1. Draw a compatibility graph in which edges mean that contracts *can* share computer time.
2. Draw a conflict graph in which the edges mean that contracts *cannot* share computer time.
3. Compare the graphs you made in Questions 1 and 2. How are they related? Which do you prefer? Why?
4. Of course, a graph doesn't tell you the number of hours of computer time needed. It just gives you another way to think about the problem. Try solving the problem by using the table and by using each of the graphs you drew. With which is the problem easiest to solve? Why?
5. As you know, in mathematical modeling it is best to start simple. Most real-world problems of this kind have more contracts. **Figure 3.3** has ten contracts instead of five. Consider the table and the two kinds of graphs you made in Questions 1 and 2. Which of the three do you prefer to use to solve this problem? (You do not have to make graphs or solve the problem unless you feel you need to do so to answer the question.)

Contract	Can share time with
A	B, C, E, G, I, J
B	A, C, D, F, G, J
C	A, B, D, F, H, I
D	B, C, F, H, I, J
E	A, F, H
F	B, C, D, E, G, H, I
G	A, B, F, I, J
H	C, D, E, F, I
I	A, C, D, G, H
J	A, B, D, F, G

Figure 3.3.
A table of ten contracts.

This lesson and this chapter are concerned with representations and procedures that can help solve optimization problems. You will recall optimization problems that occur in other chapters, such as finding an optimal location for a fire station in Chapter 1, *Gridville*, and choosing an optimal strategy in Chapter 2, *Strategies*.

Individual Work 3.1: Modeling with Graphs

Choosing a good representation is an important step in solving a problem. In this Individual Work, you practice creating graphs to represent situations. The situations are simple so that the complexity of the problem does not distract you from making good decisions about how to make a graph.

In Course 1, Chapter 0 (*Pick a Winner: Decision Making in a Democracy*), a form of graph called a runoff diagram describes runoff elections between pairs of candidates. Runoff diagrams are made from preference diagrams that show how voters rank candidates.

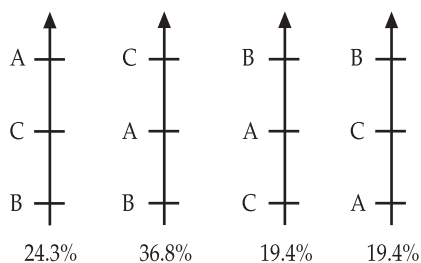


Figure 3.4.
Voter preferences.

Figure 3.4 shows preferences of voters for candidates A, B, and C. (For example, 24.3% of the voters prefer A to either B or C.) There are three ways to hold an election between two of these candidates: A versus B, B versus C, and A versus C. Suppose an election is held between A and B. If you cover C in **Figure 3.4** you see that $24.3\% + 36.8\% = 61.1\%$ of the voters prefer A to B. Therefore, A beats B in an election between A and B. To use a graph to show that A beats B:

- Draw vertices to represent A and B;
- Draw an edge between the vertices to indicate the runoff;
- Add an arrow to the edge so that it points from A to B to indicate that A beats B (**Figure 3.5**). The arrow is a useful modification because it is important to know who won the election between A and B.

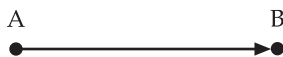


Figure 3.5.
A graph representing a runoff between candidates A and B.

A graph that uses arrows to indicate direction is called a **digraph**, which is short for directed graph.

1. Draw a digraph that shows all three runoffs. Explain how the digraph can be used to get a Condorcet winner (a candidate who can beat all the others in two-way races.)

Adding an arrow to a graph is useful when the direction of a relationship matters. Another way to modify a graph is by writing a number on each edge. For example, a graph that shows flights between cities is more useful if the costs of the flights are shown on the graph. Suppose that the cost of a flight between cities A and B is \$267, the cost of a flight between cities A and C is \$438, and the cost of a flight between cities B and C is \$193. To use a graph to show a flight between cities A and B and the cost:

- Draw two vertices for cities A and B;
- Connect the vertices with an edge;
- Write the cost along the edge (**Figure 3.6**).

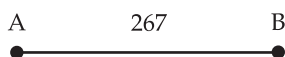


Figure 3.6.
A graph representing a flight between cities A and B.

Writing the cost by the edge is helpful if the graph is used to plan the cheapest trip.

A graph in which a number is attached to each edge is called a **weighted graph**. Note that in a weighted graph, it is not necessary to draw the edges so that their lengths are proportional to their weights.

2. Draw a weighted graph that shows all three flights.

Digraphs and weighted graphs are the most common variations of basic graphs. When you represent a situation with a graph, it is important to think about whether it is useful to attach arrows or weights to the edges. You should use arrows or weights if they are needed to solve the problem at hand.

In Questions 3–10, show at least one way to represent the situation with a graph. Be sure it is clear what the vertices and edges of the graph represent. Pay attention to the optimization problem to be solved and decide whether direction arrows or edge weights are needed. If your graph is a digraph or a weighted graph, explain why you think the arrows or weights are important.

3. A large company's headquarters is a complex of five buildings that they need to connect with fiber-optic cable. The costs of connecting buildings vary as shown in **Figure 3.7**. The company is interested in finding the cheapest way to connect the buildings.

Building	A	B	C	D	E
A		\$2500	\$1700	\$3400	\$1900
B			\$2300	\$1800	\$3100
C				\$2400	\$1600
D					\$2900
E					

Figure 3.7. Fiber-optic costs.

FYI
 Each entry in Figure 3.7 could be included twice. It is not necessary to do so since, for example, the cost of connecting A to B is the same as the cost of connecting B to A. Tables in this chapter usually do not display figures twice.

- The Swiss Mathematician Leonard Euler (pronounced “oiler”) once used a graph to solve a famous problem. The city of Königsberg in what was then Prussia (later Kaliningrad, in Russia) is divided into several parts by the Pregel River. The parts of the city were, at that time, connected by seven bridges (Figure 3.8). People tried to walk through the city, cross each bridge exactly once, and return to their starting location. (Hint: In this problem, you can have more than one edge connecting a pair of vertices.)

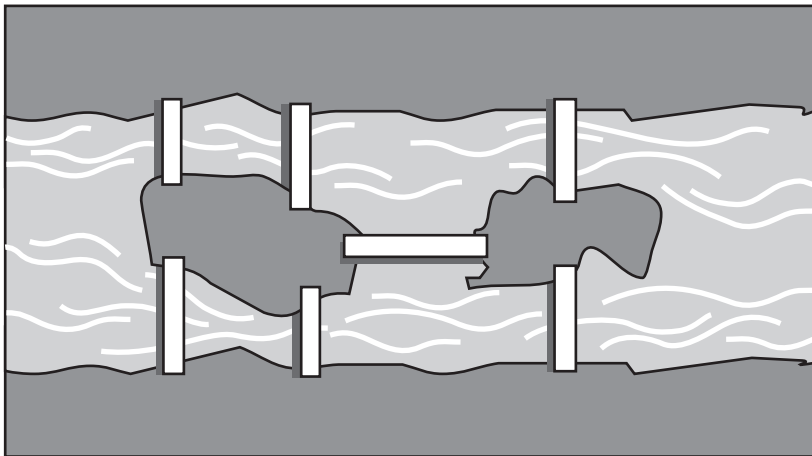


Figure 3.8. Königsberg and its seven bridges.

FYI
 Leonard Euler (1707–1783) is considered one of the greatest mathematicians of all time. He produced half of his mathematical work after going blind. Some historians believe that his work on the Königsberg bridge problem is the first use of a vertex/edge graph.

- Five teams are in a round-robin tournament, which means that each team plays every other team. When the tournament is over, team A has defeated B and D; team B has defeated C, D, and E; team C has defeated A; team D has defeated C; and team E has defeated A, C, and D. Tournament officials must decide which team receives the first-place trophy.
- Special robots are used to solder circuit boards. A robot arm travels from point to point on the board, soldering as it goes, then returns to its starting point to await the next board. **Figure 3.9** shows distances (in centimeters) between pairs of points on a circuit board. The robot must be programmed to solder the five points as quickly as possible.

Points	A	B	C	D	E
A		2.3	2.1	3.4	2.6
B			4.0	3.1	2.0
C				3.3	3.7
D					3.5
E					

Figure 3.9.
Circuit board distances.

7. A zoo is updating its enclosures from cages to large habitats that hold several species. A habitat cannot hold two species if one preys on the other. The zoo wants to find the most economical way to assign species to habitats.

- Species A preys on B, C, and E.
- Species B preys on C and D.
- Species C preys on E and F.
- Species D preys on F.
- Species E and F do not prey on any of the others.

8. A developer is building a ski resort that has four buildings. Roads must be built to connect the buildings, but it isn't necessary to build a road from every building to every other. The developer wants to build the roads as cheaply as possible. **Figure 3.10** shows costs the developer got from a road contractor.

Building	A	B	C	D
A		\$3500	\$3800	\$4200
B			\$2100	\$2400
C				\$2800
D				

Figure 3.10.
Estimated costs.



Math Used to Identify Terrorist Networks

UPI

May 7, 2009

Japanese scientists say they have used graph theory and computational data processing to create a technique that can help identify potential terrorists.

Yoshiharu Maeno and Associate Professor Yukio Ohsawa of the University of Tokyo used their new technique to analyze the terrorist network responsible for attacks on the World Trade Center and Pentagon. They said they discovered connections among terrorists that had not been apparent to security experts.

9. A manager of a rock band is planning a series of four gigs. The manager got costs of plane flights between each pair of cities in the tour and wants to buy the cheapest tickets. Of course, the tour must start and end at the group's home, which is near Chicago. **Figure 3.11** shows ticket costs between pairs of cities.

City	Chicago	St. Louis	Denver	Houston	Phoenix
Chicago		\$118	\$229	\$312	\$285
St. Louis			\$198	\$231	\$271
Denver				\$239	\$164
Houston					\$182
Phoenix					

Figure 3.11.
Plane ticket costs.

10. In most towns, there are services that travel every street: mail delivery, garbage collection, and street cleaning. People who manage these services want to cover the streets efficiently. Routes must be designed to avoid deadheading—driving back over a street that has been serviced. Draw a graph to represent the neighborhood in which you live. Explain why you think your graph is a useful model for someone planning a delivery route in your neighborhood.
11. Two people drew the graphs in **Figure 3.12** for the same situation. Could they both be right? Explain.

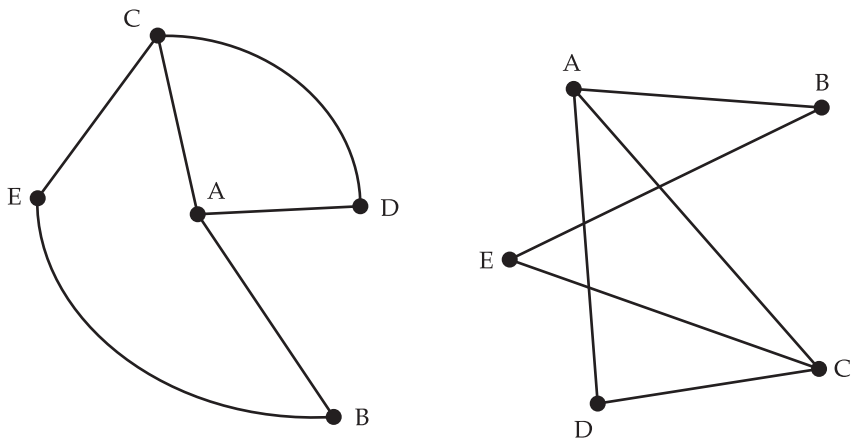


Figure 3.12.
Graphs drawn by two people.

12. In this lesson, you used graphs to model a number of situations. Although the situations are different, the graphs you drew may have led you to believe that some of the situations are alike. Go back through the situations and group together those that you think are alike in some way. Explain your groupings.