CHAPTER

6

# Growth



# GROWTH PATTERNS

Growth is part of life. Money grows in your savings account because you make deposits and earn interest. The amount of trash in a landfill grows as truckloads of garbage are added to the pile. The number of people with the flu grows as a virus spreads.

In many areas of life, people try to predict future growth based on past patterns in order to make plans for future needs. For example, if you expect your town to grow, you can plan for more schools, parks, and fire protection.

In this chapter, you model several kinds of growth, including population growth, spread of diseases, investments, use of natural resources, and drug levels in the body. You decide which family of functions gives a good model and then fine tune control numbers. The goal is to use modeling to help people make the best decisions possible.

Arithmetic sequences and series, geometric sequences and series, and mixed sequences are the mathematical core of this chapter. Tables and recursive graphs help identify patterns for several families of functions, including linear, exponential, and quadratic. Closed-form equations provide tools for making predictions. You add logarithmic functions to your tool kit of functions to assist in your work with exponential growth.

LESSON ONE

## Growing Concerns

Key Concepts

Data analysis

Additive growth

Multiplicative growth

Growth factor

Relative rate of growth

# The Right Dose

Doctors must be careful when they prescribe drugs. Too high a dose can harm the body. Too low a dose might not help the patient.

Doctors follow steps similar to those you use in mathematical modeling. In fact, the process of diagnosing and treating illness reflects modeling principles:

- Symptoms present a well-defined problem; doctors find the cause of the symptoms and treat the cause and/or provide relief.
- Doctors simplify by making a hypothesis based on the most likely causes of symptoms.
- Doctors order tests and propose drug doses based on general principles.
- Doctors test a model by analyzing blood samples. Then they adjust the dosage based on the results of the test. The process repeats until the desired equilibrium level is attained.

In this lesson, you are challenged to find a correct drug dosage for a patient. Apply what you know about functions and modeling as you investigate a new situation involving the growth of a quantity.

# Activity 6.1: Beyond Prescription

*In this activity, you begin the process of modeling the growth of medicine in a patient's body.*

**FYI**

Phenobarbital is a drug commonly used to control seizures—an anticonvulsant.

Eric suffers from seizures. His doctor prescribes phenobarbital.

- Eric takes an initial dose of 250 mg.
- He then takes 60 mg of the medicine every 12 hours.
- **Figure 6.1** show the minimum daily level of phenobarbital in Eric's blood, measured in micrograms-per-milliliter.
- The minimum therapeutic level (the amount needed to be effective) for phenobarbital is 15 mcg/ml.
- Phenobarbital levels above 40 mcg/ml can be toxic.

1. Use data analysis to predict Eric's minimum level after two weeks. Support your prediction.
2. Eric's highest level is just after each new dose and is about 1.5 mcg/ml above the preceding minimum level (which is just before each new dose). Predict a dose (taken every 12 hours) that gives a high level of 20 mcg/ml after 2 or 3 weeks.

| Days | Blood level mcg/ml |
|---|---|
| 1 | 7.5 |
| 2 | 8.5 |
| 3 | 9.4 |
| 4 | 10.0 |
| 5 | 10.5 |
| 6 | 11.0 |
| 7 | 11.3 |

**Figure 6.1.** Eric's blood levels of phenobarbital.

## DISCUSSION/REFLECTION

1. Suppose you take a single dose of a medicine. The medicine is absorbed by your body and eventually eliminated. What factors might affect the amount of the drug in your blood at a given time?
2. Describe what you think happens to the amount of medicine in your body between doses. Then give a non-medical example of a similar process.
3. As you make monthly payments on a loan, the amount you owe is affected in much the same way as the amount of medicine in your body as you take regular doses. Explain how these two systems are alike and different.
4. If you stay with an employer long enough, you expect wage raises. Comment on similarities between this and the medicine and loan examples.

# Individual Work 6.1: Moose Revisited

*In this Individual Work, you revisit tools that you used in prior chapters that will help you model growth in this chapter.*

1. **Figure 6.2** shows levels of acetaminophen in a person's blood. Compare this graph with your answer to Discussion/Reflection Question 2. Why does the level increase quickly every 6 hours? What happens between these increases?

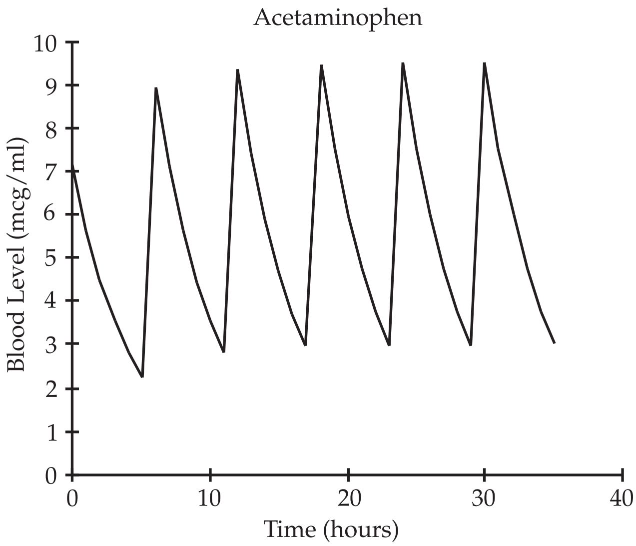


**Figure 6.2.** Acetaminophen concentration in the blood of an adult.

An **additive process** follows the pattern "next = current + constant." Additive growth is described by recursive equations of the form $p_n = p_{n-1} + k$ where $k$ is a constant.

**FYI**

Additive growth models are introduced in Course 1, Chapter 5 (*Wildlife*).

2. There are 78 moose in a park. Six new moose are added each year.

   a) Use an additive model to write a recursive equation for the total number of moose.

   b) Convert your recursive equation to a closed-form equation.

   c) Describe the appearance of a graph of total moose versus time.

   d) Predict the number of moose after 15 years.

**MODELING NOTE**

Different forms of an equation are useful for different tasks. Closed-form equations allow you to evaluate or solve in one or two steps. Recursive equations make it easy to generate a next value, to investigate rates of change, or to classify data. Often it is useful to convert an equation from one form to another.

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e) Find the total number of moose in 75 years. Explain how you found it.

f) When do you expect the moose population to reach 500? Is your answer realistic? Explain.

A **multiplicative process** follows the pattern "next = current × constant." Multiplicative growth is described by recursive equations of the form $p_n = b \times p_{n-1}$ where $b$ is a constant.

3. There are 78 moose in a park. The number increases by a factor of 1.08 each year.

   a) Use a multiplicative model to write a recursive equation for the total number of moose in the park.

   b) Convert your recursive equation to a closed-form equation.

   c) Describe the appearance of a graph of total moose versus time.

   d) Predict the total number of moose in 75 years. Explain how you got your answer.

   e) When do you expect the moose population to reach 500? Is your answer realistic? Explain.

**FYI**

Multiplicative growth models are introduced in Course 1, Chapter 5 (*Wildlife*).

Based on the recursive equation for multiplicative growth, $p_{next} = b \times p_{current}$ (or $p_n = b \times p_{n-1}$):

- $b$ is called the **growth factor**. Since $\frac{p_{next}}{p_{current}} = b$, $p_{next}$ is proportional to $p_{current}$, and $b$ is also a constant of proportionality.
- The recursive equation is sometimes written as $p_{next} = (1 + r)p_{current}$. In this case, $b = 1 + r$, and $r$ is called the **relative rate of growth**.
- The corresponding closed-form equation is $p_n = p_0 \times b^n$, an exponential function with base $b$.
- The expression $(p_{next} - p_{current})$ is the **amount of growth**—the difference between successive values in a growth sequence.
- $\frac{p_{next} - p_{current}}{p_{current}} = r$; that is, the amount of growth between successive values is proportional to $p_{current}$, and the relative rate of growth is the constant of proportionality.

**TAKE NOTE**

You can get the equation $\frac{p_{next} - p_{current}}{p_{current}} = r$ from the equation $p_{next} = (1 + r)p_{current}$ by distributing $p_{current}$, subtracting $p_{current}$, and then dividing by $p_{current}$.

4. Use a multiplicative growth model for each of these:

a) The relative rate of growth is 20%. What is the growth factor?

b) If $p_{current}$ is 640 and $p_{next}$ is 688, what is the growth factor? What is the relative rate of growth?

c) A current population is 13,500. Find the next population (one year later) if the relative rate of growth is 2.5% per year.

d) If a current population is 8340 and the relative rate of growth is 4% per year, what is the amount of growth from the current year to the next?

e) Suppose a current population is 5000 and the population in four years is 6312. What is the growth factor for one year?

f) Describe the graph of relative rate of growth versus current population.

When a multiplicative growth process results in lower populations, the process is sometimes called **decay** instead of growth. The relative decay rate is the opposite of the relative growth rate. For example, a growth rate of –5% is a decay rate of 5%. The growth factor in this case is 0.95 (= 1 + –0.05).

g) What is the relative rate of decay if the population drops from 5480 to 5314 in one period? What is the growth factor?

5. Your parents have been giving you $2.00 per week. In exchange for some chores, they agree to raise the amount by 5% each week, starting after the first week after the agreement.

a) Write a recursive equation for weekly pay. The increment of time is one week. Let $p_n$ be the pay for the $n$th week.

b) Write a closed-form equation for weekly pay. Let $n$ be the number of weeks. Let $p(n)$ be the pay after the $n$th week.

c) What is the weekly pay at the end of one year?

d) Suppose your parents offer you a choice of a $10.00 weekly increase or a 5% weekly increase. Which would you choose? Explain.

6. **Figure 6.3** shows attendance at a classic car race. The total attendance increased every year from 1993 to 2009.

a) Do you think a single function might describe these data well? How could you decide?

| Year | Attendance (millions) |
|---|---|
| 1993 | 1.6 |
| 1994 | 1.6 |
| 1995 | 1.5 |
| 1996 | 1.5 |
| 1997 | 2.1 |
| 1998 | 2.1 |
| 1999 | 2.2 |
| 2000 | 2.6 |
| 2001 | 3.0 |
| 2002 | 3.1 |
| 2003 | 3.3 |
| 2004 | 3.4 |
| 2005 | 3.7 |
| 2006 | 4.0 |
| 2007 | 4.9 |
| 2008 | 5.3 |
| 2009 | 5.6 |

**Figure 6.3.**
Automobile race attendance.

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b) How can you decide whether the data are best described by a linear, exponential, quadratic, or some other kind of function?

c) Suppose you decide the attendance pattern is linear. How do you find the best line to fit the data? How do you know if your line is the best fit?

7. For each of these, decide whether multiplicative or additive growth applies. Give reasons for your choice.

   a) Situation 1
   I started with two fish in my fish tank. A month later there were 12. The next month I counted over 70. By the end of the fourth month there were over 400. The number of fish is growing at a rate of over 50% a week.

   b) Situation 2
   The amount of water in a dam depends on how much water flows into the dam each day, how much water is released each day, and how much water evaporates each day.

   c) Situation 3
   I am reading a book. My goal is to read 25 pages each day. At this rate I expect to finish the book in about 17 days.

## *Vampires a Mathematical Impossibility, Scientist Says*

*LiveScience*

January 11, 2008

A researcher has come up with some simple math that sucks the life out of the vampire myth.

University of Central Florida physics professor Costas Efthimiou's work debunks pseudoscientific ideas in an attempt to enhance public literacy.

Efthimiou's logic: On Jan 1, 1600, the human population was 536,870,911. If the first vampire came into existence that day and bit one person a month, there would have been two vampires by Feb. 1, 1600. A month later there would have been four, and so on. In just two-and-a-half years the original human population would all have become vampires with nobody left to feed on.

If mortality rates were taken into consideration, the population would disappear much faster.

"In the long run, humans cannot survive under these conditions, even if our population were doubling each month," Efthimiou said.