

instructional units, COMAP's writers integrate mathematical skills into the broad faculty of reason, rather than insulting their readers (and denigrating mathematics itself) with toy problems set apart from real life.

Unit 1 carries the title "The Geometry of Art" -- and if you are accustomed to conventional math books, this unit may set your teeth on edge. Why does *Course 3* begin by devoting more than 100 pages to *art* projects?

The answer starts to emerge soon enough, as perspective and the vanishing point are presented as familiar visual phenomena, then as aspects of analytic geometry. Later in the unit, the writers explore complexities such as foreshortening -- and in doing this, they employ geometric proportionalities and develop trigonometric analysis in terms of sines and cosines.

The unit includes exercises in drawing and sketching, but these are not mere diversions, and they do not have anything in common with the inane amusements that one might find in, say, a math book produced by Glencoe/McGraw-Hill. The COMAP writers use exercises in drawing and sketching to promote the student's skills in making observations and expressing the mathematical relationships between geometric objects. Moreover, the ability to make sketches is a useful addition to any student's tool-set for conveying ideas. Scientists, engineers, clothing-designers, stage managers, reporters, glassblowers, landscape architects, detectives and carpenters use sketches, just as graphic artists do.

Unit 2, titled "Fairness and Apportionment," revolves around the problem of achieving fairness while dividing fixed resources among multiple claimants. How, for example, can an estate be fairly divided among three heirs if the estate consists of a car, a dog, a stereo system and some money? (Clearly, giving a third of the dog to each heir is not a solution!) That the heirs will have different preferences may seem to be a complicating factor, but it actually is the key to a successful resolution. The different preferences may even lead to a happy paradox in which every heir believes that he has got the best deal.

The centerpiece of Unit 2 is an examination of the problem of deciding how many seats in the federal House of Representatives must be awarded to each state in the Union. This problem springs directly from the Constitution's Article 1, Section 2, which dictates that a new apportionment must be conducted every ten years and must be based on population: "Representatives and direct Taxes shall be apportioned among the several States which may be included in Union, according to their respective Numbers,"

That sounds simple enough, until we remember the political fighting that erupts when, because of changes in the size and geographical distribution of the national population, a state stands to lose or gain a seat. In fact, the allocating of seats according to the states' respective populations is not straightforward at all, because there are several different ways to do this. There are different algorithms that take account of population, in one way or another, but yield different results -- and the debate over how to do the math during an apportionment reaches back to the earliest days of our republic. As the student reads on page 137 of *Course 3*, the first bill that President George Washington vetoed was a bill that involved the mathematics of apportionment.

Reading further, the student learns about, and performs some calculations with, apportionment algorithms that were advanced by Alexander Hamilton, by Thomas Jefferson, by Daniel Webster and by John Quincy Adams. The student also learns out that no method of apportionment is perfect, that every method is unfair to some extent, and that there are mathematical techniques for computing the unfairness of a specified method in a given situation. In the final lesson of Unit 2, the student uses two of those techniques for solving problems [[note 2](#)].

Wolves! -- but No Eco-Fables

Unit 3, "Sampling," deals primarily with surveys and polls -- instruments by which sociologists and journalists and bureaucrats (among others) try to assess an entire population's attitudes or preferences by questioning a relatively small group, or sample, of the population's members. The COMAP writers begin the unit by discussing a real event: a survey that was mounted in 1990 to appraise public opinion about a proposal to restore the gray wolf to the Upper Peninsula of Michigan.

If you are familiar with ordinary math books, what I have just written has probably made you brace yourself. If an ordinary math book were to mention wolves at all, the wolves doubtless would become characters in fuzzy-animal stories, emotional eco-fables, or inspirational "balance of nature" myths. In COMAP's book, this never happens. COMAP's writers not only avoid emotional writing; they teach the student how words and phrases that excite emotions can influence the responses that people give during opinion polls. In fact, in an exercise on page 191, the writers direct the student to "Make up a survey question or series of questions that includes one or more emotionally-charged [terms] designed to bias responses toward a particular viewpoint." The purpose of this exercise is to help the student to recognize and understand sources of bias, with the larger goal of helping him to minimize bias in his own work.

As the unit continues, the student and his classmates write questionnaires, respond to questionnaires, and analyze the results. This leads to a discussion of sampling uncertainty and to an exceptionally good introduction to the concept of the statistical confidence interval.

The last lesson in Unit 3 is devoted to sampling in a different context. The challenge now is to learn the size of a population of wild animals by using the mark-and-recapture method [[note 3](#)]. The same mathematical techniques that the student already has employed for analyzing the results of surveys can be invoked for analyzing mark-and-recapture data, so the COMAP writers present a series of exercises in which the student estimates the sizes of various populations, calculates confidence intervals, and considers sources of bias. Here the sources of bias are not loaded words or loaded questions; they are circumstances or accidents that may introduce errors into mark-and-recapture studies or violate the assumptions on which mark-and-recapture studies are based [[note 4](#)]. This lesson provides a fine example of how COMAP's writers teach that the same set of mathematical tools can be used in very different contexts and can be applied to very different kinds of problems.

Going into Business

Unit 4, "Mind Your Own Business," is aimed at showing the student "how mathematical models can help small businesses develop strategies to ensure their growth and prosperity." It starts with two broad questions for the student to keep in mind: "How is the success of a small business measured?" and "What factors affect the success of a small business?" Then the student moves to developing mathematical models that incorporate things like loan interest, the value of finished goods on hand, taxes, utility costs, money in the bank, and limits on production and storage capacity. The scope of the model expands as the COMAP writers introduce such concepts as marginal cost, marginal revenue, and price elasticity. I have seen educated adults stumble when they had to deal with those concepts.

Lesson Two in this unit starts with an activity titled "The Cost of Money," meaning both the interest that a business must pay to borrow money and the interest that the business will lose if it withdraws its own money

from an interest-bearing account. After this concept has been introduced, every balance sheet takes account of the cost of money. I never saw material like this when I was taking courses in high school. National data on personal debt and personal bankruptcies suggest that lots of other people did not see it either.

Unit 5, titled "Oscillation," deals with cyclic phenomena, such as trajectories of bouncing balls, day-by-day measurements of the number of hours of daylight, and month-by-month reports of housing starts. The material in this unit is especially effective in reminding the reader that -- as COMAP's writers promised in their introductory statement -- real problems are messy. Yes, housing starts and electricity bills and water bills show seasonal variations, but the data reflect plenty of random variation too. On page 484 a table that shows a community's monthly consumption of water for four years reinforces the messiness lesson by presenting one value that is conspicuously anomalous and doesn't make sense. The COMAP writers don't mention this outlying value in their text, but there it is -- and the student will have to deal with it in one way or another.

Of course, any discussion of cyclic behavior eventually turns to sines and cosines, but the interpretation of sines and cosines in Unit 5 is decidedly different from the geometric view of sines and cosines that prevailed in Unit 1.

In the fourth lesson of this unit, the writers address sound as a cyclic phenomenon, and the student utilizes a microphone and an electronic digitizer to convert sounds into signals that can be fed directly to a graphing calculator. This hands-on approach to science and math leads naturally to discussions of superimposed sines and damped sinusoids. Before the unit ends, the student has plotted cyclic functions, has used sines to approximate cyclic functions, and has computed the residual errors between true functions and the sine-approximations of those functions. The student then fits other sines to the residuals and repeats the process.

Realistic Expectations

In *Course 3*, as in *Course 1* and *Course 2*, the writers often hint at mathematics that the student will encounter in future years. Here in Unit 5, for example, the writers introduce Fourier analysis without actually using that term. Elsewhere in the unit, they ask questions that the student will not truly be able to answer until he has learned some calculus. The writers implicitly accept the fact that a high-school student will not have a mathematical tool for producing a rigorous solution to every problem. They expect only that the student will get the best answer that can be generated with the mathematical techniques that are at hand.

Unit 6 nominally addresses "Feedback" but actually covers a lot more. It focuses on situations in which multiple phenomena interact, with a graphical notation for telling how each factor enhances or suppresses the others. One example of a multiple-phenomenon interaction is the spread of an infectious disease: As the number of infected individuals increases, so does the number of new cases of infection, up to some point -- but then the occurrence of new infections tapers off, as fewer individuals are left to be infected. The same model applies to the propagation of a new joke (page 529) or to sales of a new product whose commercial fortunes depend on word-of-mouth reports by consumers.

Along the way, COMAP's writers introduce advanced topics in unthreatening ways, as when they employ phase-plane diagrams to show how multiple effects interact over time (page 548). This yields a happy surprise -- a way to view the classic predator-prey problem in which a population of rabbits grows or shrinks according to the size of the local lynx population, while the lynx population grows or crashes according to the availability of rabbits to eat. Students normally encounter the phase-plane analysis of a predator-prey system after a year or two of calculus, but the basic concepts and analysis are fully appropriate for students who have been using *Course 3*.

The writers proffer another foretaste of advanced analysis in exercise 3d on page 537. They propose two models for explaining a system's behavior, then invite the student to choose between them. This carries the flavor of what statisticians call Bayesian analysis. In traditional statistics, one assumes some model and asks, "Given my model, how well do my data match it?" In Bayesian analysis, that question is turned around and becomes "Given my data, how well does this model explain them?" This makes it possible to quantify and compare the explanatory power of different theoretic models in the presence of real-world data.

The writers falter only once in Unit 6. On page 501, after telling that a skydiver experiences increasing air resistance with increasing downward speed, they say: "So, if air resistance could be made large enough, then it could slow the skydiver, but speed alone is not capable of causing that large an increase in resistance." As a skydiver who has made more than 800 jumps, I can tell you that this understates the magnitude and importance of air resistance. Under some circumstances, air resistance can in fact slow a skydiver down. Moreover, it normally abolishes his downward acceleration. When a human is in free fall, the air resistance that he experiences increases rapidly with speed, and it normally becomes as great as his weight after ten to twelve seconds. When those two forces are equal, the skydiver still has a downward velocity, but his downward acceleration goes to zero.

Putting Mathematics to Work

Unit 7 is called "Modeling Your World." The COMAP writers evidently created it with this point in mind: Mathematical material is not really mastered until it is put to use. Accordingly, they now ask the student to conceive and carry out a project based on what he has learned, and to describe the project in a report. To show what is expected here, they provide four models -- four reports in which students have told about what they have done and the results that they have obtained.

Good textbook-writers choose subjects that will engage their readers, so the first report deals with acquiring an expensive, "loaded" car. The student has done a clear-eyed analysis of whether he should buy the car with cash from his bank account, take a loan to cover most of the purchase price, or lease the car. After considering such factors as equity, the interest that he would have to pay for a loan, and the interest that he would sacrifice by withdrawing money from his bank account, he has concluded that it will be advantageous to get the car by leasing it.

The second and third of the model reports tell about projects that dealt with historical issues: fairness in American elections, and an 18th-century war in which a royal succession was at stake.

The fourth report is, to me, the most interesting because it deals with an undertaking that failed. It describes an effort to automate the process of laying out the pages of a newspaper, given rules about the placement of advertisements and the permissible ways to break individual articles across pages. A high-school-level treatment of this problem failed to generate any acceptable algorithm. Still, there is as much valuable content here as in the other model reports, and a lot more intellectual bravery.

A common problem in teaching is that different classes run through their curriculum at slightly different rates, so the teacher may face a shortage of time or an excess of time as the school year approaches its end. Unit 7 can easily be shortened or extended to compensate for either situation, so it can act as a buffer. Having been a teacher, I appreciate this feature of Unit 7 immensely.

Course 3, like the other COMAP books, encourages intelligent criticism and self-criticism. The writers ask the student to evaluate his own work, to check his work against that of other students, to question the assumptions behind analyses, to consider the effects of broken assumptions, and to use sensitivity analysis for finding out how a calculated result would change if the inputs were a bit different. And, very importantly, they ask the student to revisit some of his early work after he has gained more knowledge. This isn't a "Gotcha!" stunt. The COMAP writers are showing students that it is healthy habit of mind to develop new answers that take account of new information. Compare this with the feel-goodism and inane self-esteemery that we see in other math textbooks.

No textbook can be perfect, and I have noticed three flaws in *Course 3*. One of these is the poor handling of the topic of air resistance, in Unit 6. Here are the two others:

- While this book generally wins our respect by avoiding political correctness, it does descend to displaying a few perfunctory photographs whose only evident purpose is to show women, members of racial minorities, and a paraplegic.
- On pages 413 through 416, the COMAP writers treat as a fact a silly superstition taken from *The Old Farmer's 1997 Almanac* -- "The best time to plant flowers and vegetables that bear crops above ground is during the *light* of the Moon; that is, between the day the Moon is new to the day it is full. Flowering bulbs and vegetables that bear crops below ground should be planted during the *dark* of the Moon; that is from the day after it is full to the day before it is new again." The writers endorse this pseudoscience by presenting an exercise in which the student, after consulting a graph that shows how much of the Moon is illuminated on each of the first 100 days of the year, determines the "best planting dates" for carrots and for flowers.

These few faults, however, do very little damage to such a fine book.


I recommend *Course 3* and the entire *COMAP's Mathematics: Modeling Our World* series to high-school students and to their teachers.

Notes

1. See my [review of Course 1](#) in *The Textbook Letter*, Vol. 11, No. 5, and my [review of Course 2](#) in Vol. 12, No. 3. [[return to text](#)]
2. The current method for allocating seats in the House was adopted in 1941. It was advanced not by a statesman but by Joseph A. Hill (who worked as a statistician for the Bureau of the Census) and Edward V. Huntington (who was a professor of mathematics at Harvard). This method is described in a box on page 163 of *Course 3*. [[return to text](#)]
3. In a typical mark-and-recapture study, harmless traps are used for capturing a sample of the population, and the individuals in this sample are marked and released; then, after some time has elapsed, the traps are used for capturing another sample. The ratio of marked to unmarked individuals in this second sample can be plugged into an algorithm that yields an estimate of the size of the overall population. [[return to text](#)]

4. One of these is the assumption that marking an individual does not affect its vulnerability to predators. Another is the assumption that trapping does not make an individual "trap-shy" and thus reduce the likelihood that the individual will be recaptured. [[return to text](#)]

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