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There is a great deal of sturm and drung about the Common Core State Standards in Mathematics (CCSSM) these days. Almost all of the conversation is political in one way or another. Critics on the right fear that it represents a federalization of U.S. education. Critics on the left fear that it represents an attack on teachers, as evaluation of teacher performance is being tied to student performance on high-stakes tests.

There is also a great deal of talk about the difficulties of implementing the standards in the time allowed. Teachers aren’t ready to teach to them. Students aren’t prepared for the testing associated with them. This is a paradigm shift that should take decades and is upon us before we are ready. Much of this is worth talking about. But we have taken our eyes off the ball.

When CCSSM first came out the discussion was all about what was in them, i.e. the content of the standards. Was this the mathematics we should be teaching? Are the practices being truly incorporated in a natural way? How can we make CCSSM into a living document, so that we can make changes when we see the need for change? These were the questions we were all asking.

But we no longer hear what I argue are the truly important questions above the noise of the politics of implementation. These fights will go on with likely no real winner. Some version of the CCSSM and the tests being devised will survive the clamor. But we need to remind ourselves, and all who will listen, that it is the substance of these standards that are important. And it is our job to be sure that we never lose sight of the fact that at heart what counts is what we teach and how we teach it.

Sol Garfunkel

CONSORTIUM
From the Publisher’s Desk

EYE ON THE BALL

CONSORTIUM

Publisher: Solomon Garfunkel
Editor: Gary Froelich

Department Editors

HiMCM Notes: William P. Fox
Math Today: Paul Kehle
Joseph Malkevitch
Jon D. Davis
Jonathan Choate
Henry Pollak
Sarah Williams
HiMAP Pull-Out Section: Marsha Davis

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When Synta Bogan goes to work, she summarizes and interprets data to help run education programs for kids. She is a financial analyst.

When Larry Cao goes to work, he models shapes and motion with mathematical equations to build apps that help us run our lives from our phones. He is a software developer.

Financial analyst and software developer were two of the top careers of 2014, according to online money magazine Forbes.com. Like dozens of news outlets, Forbes crunches government and private jobs data and announces its own “Top 10” list. Forbes’s list awards points for past and projected growth and for wages above $22 per hour.

Students can create their own career ranking system, by accessing the government data directly at www.bls.gov/ooh, the Bureau of Labor Statistics’ Occupational Outlook Handbook website. Along with descriptions of the work environment and valued skills in each occupation, the government numbers report salaries (radiation therapists: $77,560 median pay in 2012), numbers of jobs (mathematicians: 3,500 in 2012), and a projection of whether jobs are being added (physical therapists: 73,500 new U.S. jobs are predicted by 2022) or reduced (airline and commercial pilots: 800 fewer U.S. jobs are predicted by 2022).

Questions for further thought:

- According to www.bls.gov/ooh, what are the education requirements for jobs you’re interested in?
- In creating your personal “Top 10” careers list, does pay matter? How about the number of jobs?

Software Developer: Building New Technologies

The top job of 2014 was Software Developer, according to Forbes.com. These folks program and tweak all the new apps we love to play with, and they design and build programs that help businesses run smoothly in the modern world.
Larry Cao, a Computer Science major at the University of California, Berkeley, has landed software internships at Nest Labs (maker of smart home devices, purchased by Google for $3.2 billion) and Twitter (transmitter of social revolution, celebrity gossip, and everything in between).

“What I like about the field,” says Larry, “is the freedom and creativity that come along with the work.” Dreaming up an elegant design to accomplish a work assignment is “almost like a puzzle or challenge.”

I asked Larry about the relationship between math and his work. “I think that math skills are crucial,” he says. “Even writing basic applications sometimes requires knowledge of math and physics. For example, animating objects like rain or snow at a certain degree requires use of trigonometry (I wrote the weather animations on the Nest app for iPhone).”

“Spring physics are also very widely used in user interface animations these days. More advanced computer graphics requires in-depth knowledge of linear algebra as well.”

“I did have to take a few math classes to declare my major, such as discrete mathematics and probability theory, linear algebra and differential equations, and two semesters of calculus.”

In his internships, Larry has also been exposed to research-level uses of mathematics in consumer computing. “Statistics and probability are becoming very important for use cases such as spam filtering or detecting fraud, and even suggesting new music to listen to. At twitter, they have teams of data scientists (most with a Ph.D. in math or statistics) that analyze petabytes of data using statistics and mathematical models.”

“This is really just scratching the surface of what is possible with math and software, but these are some of things that I have come across so far.”

The most exciting part of the job? “Possibly having your work used or seen by millions of people.”

Financial Analyst: Helping Organizations Thrive

Financial Analyst was the #4 job of 2014, says Forbes.com. These folks scout out the money landscape and report back to their teammates. The analyst’s summaries and graphical snapshots convey the important information necessary for decision-making about investments and budgeting.

Synta Bogan is a financial analyst at the University of California, Berkeley, in the Center for Educational Partnerships. Her organization helps students facing “significant barriers” attend college, offering mentoring, counseling, and college prep enrichment programs in schools and on the flagship Berkeley campus.

Although her workplace focuses on education and social issues, to Synta’s financial mind, the organization is still very much a business. She explains, “I like that I can manipulate the numbers into information that is useful to run a business and to project for the future of a business.”

Synta took classes in trigonometry and calculus in high school (“because I had to,” she admits), but, she says, her interest in numbers developed outside of the classroom. “I was always good with fine details and Microsoft Excel, so I parlayed those skills into bookkeeping and project management positions. Through bookkeeping and project...
management, I learned how to read, manage and project budgets.”

Synta’s college majors were sociology and education, not accounting or business. She explains, “I learned how to ‘analyze’ finances by mostly on-the-job training, but have taken several accounting courses to formalize my knowledge.” She advises students interested in this career to get involved both in and out of school. She says students can “take accounting classes, and get hands-on experience by managing their own life budgets, as well as budgets for a business or project. Pay attention to the details and check your formulas for accuracy. Learn Excel and other accounting software, like QuickBooks.”

The O*NET OnLine website, a free resource for people exploring careers, emphasizes the importance of clear communication in this profession. Financial analysts “draw charts and graphs, using computer spreadsheets, to illustrate technical reports,” and also “present oral or written reports.” Synta agrees that it’s her job to translate data into information that people are ready to use. “Reviewing budgets can be daunting for a lot of people, so I format the budgets in a way that is easily understandable for my colleagues and clients.”

Synta’s reporting helps “people see and understand their spending habits, where their money goes, and how to plan better for the future.” Her final word of advice for aspiring analysts? “Don’t make your reports too complicated; many people don’t see numbers the same as you!”

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**Going Further With Jobs Data: Location Quotient**

For top careers, and for the job that speaks to you, location matters. **Location Quotient**, a calculator link found at the bottom of the bls.gov website, leads to even more information on jobs. But what does it mean?

As an example, consider two industries and two counties in Oregon. Like many states, Oregon has regions that rely on older industries for jobs, as well as high tech corridors. By glancing at the Location Quotients in the top half of this table, can you tell which Oregon county is in each category?

Further down the table, are you surprised to find that the number of wood products jobs in each county is similar, falling in the range 1,000-2,000? Are both parts of the table informative when thinking about careers?

Here is the definition of Location Quotient (LQ), from the Bureau of Labor Statistics:

“LQs are calculated by first dividing local industry employment by the all industry total of local employment. Second, [U.S.] industry employment is divided by the all industry total for the [U.S.]. Finally, the local ratio is divided by the [U.S.] ratio.”

To check the LQ numbers in the table, you’ll need to know the total U.S. jobs (112,948,842), and the U.S. jobs in wood products manufacturing (353,678) and computer manufacturing (1,061,520). Questions for further thought:

- What is the meaning of each ratio described below?
- Could they be expressed as percentages?
- How would you choose to round these numbers?
- In creating your personal “Top 10” list of careers, does LQ matter?

**Comparison of Jobs in two Oregon Counties**

<table>
<thead>
<tr>
<th>Location Quotient</th>
<th>Jackson County</th>
<th>Washington County</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wood product manufacturing</td>
<td>8.65</td>
<td>1.38</td>
</tr>
<tr>
<td>Computer and electronic product manufacturing</td>
<td>0.56</td>
<td>12.04</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Number of Jobs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wood product manufacturing</td>
</tr>
<tr>
<td>Computer and electronic product manufacturing</td>
</tr>
<tr>
<td>All industries</td>
</tr>
</tbody>
</table>
Resources


Sarah Williams teaches math at Foothill College in California’s Silicon Valley. Reach her at WilliamsSarah@Foothill.edu
The Relationship Among Teacher Knowledge, Teacher Perceptions, and Student Achievement

Jon D. Davis

It seems natural that teachers’ knowledge should affect what students learn in mathematics classrooms. Indeed, research suggests that this is the case for high school mathematics teachers (Rice, 2003) and elementary teachers at grades 1 and 3 (Hill, Rowan, & Ball, 2005). Recently, a study conducted in Germany established that both content knowledge and pedagogical content knowledge in the area of mathematics were significant predictors of student achievement (Baumert et al., 2010). They also assert that pedagogical content knowledge was the stronger predictor of student achievement. A study recently published in Journal for Research in Mathematics Education by Campbell et al. (2014) makes an important research contribution by exploring the connection between mathematics teacher factors and student understanding. Specifically, they examined the influence of teacher beliefs and teacher knowledge (content knowledge and pedagogical content knowledge) on student achievement as measured by standardized test scores. In the paragraphs that follow I will describe the methods used in this study, its results, and its implications.

METHODS

All teachers voluntarily elected to participate so there was no randomization in use during this study. A total of 259 upper-elementary (grades 4 and 5) and 184 middle-grades (grades 6, 7, and 8) teachers with 6 or fewer years of experience were enrolled in the study. These teachers came from 23 school districts located in Delaware, Maryland, and Pennsylvania. Teachers completed a test of teacher knowledge and surveys examining their perceptions, professional background, and instructional context. At the upper-elementary level, the teachers worked with a total of 6,413 students and at the middle-grades level there were a total of 10,890 students.

Two assessments of teacher knowledge were administered to both the upper-elementary and the middle-grades mathematics teachers. One set of assessments measured content knowledge and the other assessment measured pedagogical content knowledge. These questions had good empirical reliability values of at least 0.704. Content knowledge assessments for upper-elementary and middle-grades teachers measured teachers’ understanding in the following areas: number and operations; geometry; measurement; probability; and data analysis; patterns, functions, and algebra. The content knowledge tests for both groups of teachers contained a total of 80 items. Twenty-five items in each test involved recall items (e.g., definitions), forty items in each test involved skill and concept items (e.g., multistep problems), and fifteen items were dedicated to strategic thinking items (e.g., making conceptions among mathematical concepts).

The pedagogical content knowledge assessment contained 40 items across four domains and six mathematical content areas: number and operations; geometry; measurement; probability; data analysis; and patterns, functions, and algebra. The four domains and number of questions in each are as follows: common student errors and misconceptions (16 items); mathematical representations and contexts (5 items); sense of order for mathematical content (5 items for upper-elementary and 4 items for middle-grades); and addressing and understanding students’ interpretations of mathematics (14 items for upper-elementary and 15 items for middle-grades).

The belief survey was constructed from an earlier survey and adapted to be more contemporary. It measured five different categories: focus of mathematics classroom instruction; how instruction should be ordered and classroom and materials should be organized; how students learn mathematics best; the role of students in the mathematics classroom; and the role of the teacher in the mathematics classroom. A factor analysis was conducted on the responses of 459 teachers. Three factors appeared in the belief survey. The first factor was related to student struggle. In other words, teachers agreeing to questions comprising this factor believed that students should struggle in solving
problems before teacher intervention. The second factor involved incremental mastery of procedural skills before engaging in problem solving. The third factor was related to student dispositions. These survey questions were connected to the belief that teachers should know students' mathematical dispositions and include problems with multiple approaches.

Educational systems are nested structures. That is, students are situated within classrooms, which are in turn situated within schools and so on. Due to this nested nature the researchers employed a two-level hierarchical linear model within the study design. The first level consisted of students with the second level consisting of teachers. At the student level, demographic variables such as gender, special education, poverty, English-language-learner status, and prior student achievement were included. For upper-elementary teachers the following variables were used in the models: teacher content knowledge, perceptions, professional background and teaching assignment (e.g., math only upper-elementary teacher), and the interactions between teacher content knowledge and perceptions. All of these predictors were used for middle-grades teachers with the exception of teaching assignment as all of these teachers were teaching mathematics. A total of three different models were fitted for both groups of teachers with three different teacher knowledge variables: content knowledge, pedagogical content knowledge, and a combination of pedagogical content knowledge and content knowledge. Due to the similarities in results between the knowledge combination model and the content knowledge model only results for the first two models will be discussed in this article.

RESULTS

CONTENT KNOWLEDGE INFLUENCES

There was a statistically significant positive relationship between teachers' (upper-elementary and middle-grades) content knowledge and students' standardized mathematics test scores. This effect was somewhat lower at the upper-elementary than in the middle-grades. At the lower-elementary level, for each increase in standard deviation teachers' content knowledge students' scores increased by 7.1% ($p = .033$). At the middle-grades level, for each increase in standard deviation teachers' content knowledge students' scores increased by 16.6% ($p < .001$). If the model was limited to student demographic variables and student prior achievement, middle-grades teachers' content knowledge was a significant predictor of students' standardized achievement scores, however, content knowledge was not a significant predictor of student achievement when a similar model was constructed at the upper-elementary level. The relationship between upper-elementary teachers' content knowledge and student achievement did not appear until other teacher variables such as professional background were included in the model.

There was a positive association between student achievement and upper-elementary teachers' stated awareness of students' mathematical dispositions, but this relationship was not statistically significant ($p = .053$). The interaction between content knowledge and awareness was statistically significant ($p = .014$) meaning that awareness influenced the effect of content knowledge on student achievement for upper-elementary teachers. That is, upper-elementary teachers' with high levels of content knowledge and high levels of awareness of students' mathematical dispositions tended to have students with higher standardized achievement test scores than teachers with high content knowledge and low awareness scores.

There was a very interesting relationship among middle-grades teachers' mathematics content knowledge, incremental mastery of procedural skills, and student achievement on standardized tests. Middle-grades teachers with low content knowledge and high scores on the incremental mastery of procedural skills believed had lower student achievement test scores than middle-grades teachers with low content knowledge and low scores on the incremental mastery of procedural skills belief. For high content knowledge middle-grades teachers this relationship was reversed. That is, teachers with high content knowledge and high incremental mastery skills taught students with high standardized test scores while high content knowledge teachers with low incremental mastery worked with students with lower test scores.

PEDAGOGICAL CONTENT INFLUENCES

Pedagogical content knowledge (PCK) was not a significant predictor of students' standardized test scores at the upper-elementary level. However, PCK was a strong predictor of student achievement at the middle-grades level. In fact, for each standard deviation increase in middle-grades teachers' PCK, students' mathematics achievement increased 22.1% ($p < .001$). At the upper-elementary level, student factors such as poverty, special-education status, and prior achievement were significantly related to student achievement. The only teacher level variable related to student achievement at the upper-elementary level was instructional assignment. That is, teachers who taught content that was deemed to be at a level higher for at least some time during the school year had students...
who tested higher on standardized achievement test scores \((p < .01)\). Interestingly, middle-grades teachers who possessed special education certification tended to work with students who had lower student achievement test scores when other student level factors were taken into account \((p < .001)\). There was also a stronger relationship between PCK and the belief in incremental mastery for middle-school mathematics teachers than between content knowledge and this belief. Students learning in classrooms with teachers who had low PCK and high incremental mastery tended to do less well than students learning in classrooms with middle-school mathematics teachers with low PCK and low incremental mastery scores.

Teaching experience was not a statistically significant predictor of student achievement at the upper-elementary level. Experience was an important predictor of student achievement at the middle-grades level. Specifically, for each year of teaching experience that middle-grades teachers possessed their students tended to score 4.9% of a standard deviation higher on the standardized achievement test \((p = .047)\).

**DISCUSSION**

For upper-elementary teachers, pedagogical content knowledge was not a statistically significant predictor of students’ standardized test achievement and content knowledge, while statistically significant, played a small role in student achievement. This finding might be due to the data collected and included in the model. In other words, other student, teacher, or school factors not included or considered in this study might be influencing student achievement at the upper-elementary level.

The National Mathematics Advisory Panel (2008) recommended that elementary level students learn mathematics from mathematics specialists. A number of teachers in this study were only teaching mathematics and hence could be considered elementary mathematics specialists. However, this teacher factor was not a significant predictor of students’ mathematics achievement in any of the models used in this study. As the authors of the study note, “Simply” implementing a policy that focuses responsibility for upper-elementary students’ mathematics instruction on fewer teachers will not ensure that those teachers have a deeper knowledge of mathematics content and pedagogy, nor will it ensure that they are the most effective teachers of mathematics available, even if they are willing to be specialized mathematics teachers in the upper-elementary grades” (Campbell et al., 2014, p. 452).

The results of the study highlight the importance of content knowledge and pedagogical content knowledge for middle-grades teachers. For each standard deviation increase in content knowledge or pedagogical content knowledge students’ standardized test scores increased by 22% of a standard deviation. This finding suggests that targeting these areas of teachers’ knowledge should be an important component of professional development and if other efforts to improve middle-grades mathematics instruction don’t target these knowledge areas, they may be hindered.

As mentioned earlier, upper-elementary and middle-grade teachers who held special education certifications taught students who exhibited lower standardized test scores. This was not due to the fact that these teachers tend to work with students with lower prior achievement. This student level variable was controlled for in the models used in the study. Rather, it suggests that there are differences in how these teachers’ content and pedagogical content knowledge influences student achievement on standardized tests. Neild, Farely-Ripple, and Byrnes (2009) found similar differences in student achievement between middle-grades mathematics teachers and middle-grades teachers with special education certification. Professional development efforts might be designed around this group of teachers.

This study confirms a long-held suspicion that teachers’ beliefs influence student achievement. For instance, this study found that teachers with high content knowledge typically have higher student achievement if they are open to and perceive their students’ mathematical dispositions. This study found this relationship to exist, but not to explain why this relationship predicts higher student achievement. The study authors pose the question, “Does heightened awareness of their students’ mathematical dispositions lead these teachers to consider whether their instructional practices are actually promoting understanding for all of their students rather than presuming the effectiveness of their methods?” (Campbell et al., 2014, p. 454)

An interaction between teacher knowledge (content and pedagogical content knowledge) and beliefs about modeling for incremental mastery was statistically significant. If teachers had weak knowledge, high scores in this belief were connected with low student achievement while strong teacher knowledge and high belief scores were related to high student achievement. This suggests that providing students with incremental mastery (demonstrate a procedure, guided practice with a procedure, followed by independent practice) will not typically yield high student achievement if teachers’ knowledge is weak. Additionally, providing
procedural scripts to teachers in the form of curriculum materials may not yield high student achievement for teachers with weak knowledge.

The authors note that measures for teachers’ content knowledge and student achievement were modeled on state curriculum frameworks and not on the Common Core State Standards for Mathematics (CCSSM), which now permeate the mathematics education environment in the United States. Nonetheless, the findings of the study with regard to content knowledge, pedagogical content knowledge, beliefs, and student achievement are even more pertinent in the CCSSM environment. This follows because the standards for mathematics practice in CCSSM emphasize a set of beliefs that teachers may struggle to enact. In addition, research involving the CCSSM suggests that teachers will need to deepen their current understandings of mathematics (Sztajn, Marrongelle, & Smith, 2011).

CONCLUSION

The study by Campbell et al. (2014) highlight the important role of teachers’ beliefs and knowledge in students’ achievement on standardized tests. While the construction of standards, tests connected to standards, and teacher evaluation connecting teachers to CCSSM have been foci of federal educational policy, it remains to be seen if policy will be developed and money provided for professional development focusing on teacher knowledge needed to implement CCSSM and teacher beliefs aligned with the standards for mathematical practice. The effectiveness of CCSSM will most certainly hinge on whether teachers are supported in implementing these standards.

REFERENCES


Jon D. Davis is an Associate Professor of Mathematics Education at Western Michigan University. His research interests include student achievement and teacher interactions within reform-oriented mathematics curricula, the use of computer algebra systems in the written and enacted curriculum, and reasoning and proof.
WORDS

JOSEPH MALKEVITCH

When people think about mathematics they probably first think of numbers and not words. But mathematics, with its great intellectual curiosity for ideas, studies words from many points of view—doing what it often does, looking for patterns. In fact, the “science” of patterns is sometimes offered as a succinct definition for the subject matter of mathematics as a whole.

If a browser is “fed” a string of words, it “returns” a list of prioritized web items tailored to that string. How is this feat accomplished? It might be that someone designing software to check if a particular essay has been plagiarized might turn to mathematics for assistance. Words interest molecular biologists who see the genetics of different species “spelled” out using the letters C, A, T, G which constitute the DNA alphabet. One goal is to determine genes in contrast to other DNA, although some DNA in the past sometimes referred to as junk DNA is now known to have biological functions. If one had insights into “words” as objects onto themselves, perhaps it might help with getting insights into these situations. Here I will take a brief look at some of the ways mathematics has looked at words in the hope of exploiting the resulting ideas.

Fibonacci numbers and words

When you see numbers like 14523, 32145, and 11223 there are various patterns you might notice and “facts” associated with the numbers. The first two have all distinct digits. For 11223 the digits used include no digits beyond those used in the first two. All of these numbers have 5 digits and the second one can’t be prime. (Primes are those positive integers bigger than 1 whose only factors are 1 and themselves.) Implicit in this discussion is that we are looking at numbers base 10, with the digits 0, 1, ..., 9. However, there is another point of view about these “numbers.” They can be viewed as strings or “words.” When we in America think about words, we usually think about English words like scared, sacred, scarred and cedars. These words have different meaning in English but they all use the same letters.

When mathematicians think about words they don’t think about what the words mean in a particular language like English or French, they think about issues related to patterns.

Look at this sequence of numbers:
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ..., (*)
and this sequence of letters:
a, ab, aba, abaab, abaababaab, (**) Note that all the strings of the sequence (**) begin with the letter a. For the first sequence (*) we get from one term to the next in the sequence by thinking of a numerical pattern. The next number in the sequence is generated by adding the two previous terms of the sequence (list):

\[ F_n = F_{n-1} + F_{n-2} \]

We started sequence (*) with \( n = 0 \) so the formula above is valid for \( n \) equal to 2 or greater, and \( F_0 = 0 \) and \( F_1 = 1 \). Sometimes it is convenient to start this sequence 1, 1, 2, ..., rather than has been done here. In addition to the difference or recursion equation (computes later values in a sequence in terms of prior ones) given above, we need “initial” values or “conditions” for this recursion equation to “act” on. Note that because integers commute, the order in which a consecutive pair is added does not matter. We could have written the “recursion” (difference equation) below instead and we would get the same sequence of integers as we see in (*):

\[ F_n = F_{n-2} + F_{n-1} \]

In the sequence (**), we generated a sequence of strings, and perhaps you can see the analogy between (*) and generating strings from prior strings in (**). To get the next string in (**) we take the prior string, write it down, and then follow it with the string two terms back. Thus, we can use the notation of recursion (difference equations) to generate a new string from a previous pair of strings by:

\[ f_n = f_{n-1}f_{n-2} \]
If we apply this recursion to the initial sequence of strings: \(a, ab\) (where \(n\) is at least 2 and treat \(f_0 = a\) and \(f_1 = ab\)) we get the sequence:

\[
a, ab, aba, abaab, abaabaabaab, ...
\]

The “fancy” term is that we concatenated two previous strings to get the next string. We needed initial conditions on which the recursion equation can act. It is traditional to denote concatenation using the symbol for “multiplication,” as we have done above. Note that the lengths of these words are 1, 2, 3, 5, 8, 13, ..., which are the numbers of the Fibonacci number sequence (starting with the third term since our pattern begins with the second term). The numbers \(0, 1, 2, 3, 5, 8, 13, ...\) which are called the Fibonacci numbers are the result of the equation:

\[
f_n = f_{n-1} + f_{n-2}
\]

The theorem applied not only to the particular two-symbol alphabet.

Do something like this in binary, writing down the integers represented in binary, gives rise to what is called the Champernowne word:

\[
01011001110101110000....
\]

which has been studied for its interesting properties.

Less strange perhaps is to form this infinite string based on the entries of the Fibonacci word sequence (**):

\[
\text{abaababaabaabaabaabaabaabaabaabaabaaba...}
\]

What we have done is to create a string where each of \(f_1, f_2, f_3, ...\) appears as an initial segment or prefix of the infinite string above. There is a sense in which this infinite string can be interpreted as a “limit” of the sequence of words in somewhat the same way that an infinite sequence of rational numbers can have a real number as its limit.

The human vision system is remarkable in being able to pick out patterns. Look at the infinite Fibonacci word above. You probably notice that there are lots of places where the pattern \(aa\) appears. However, do you see any occurrences of the pattern \(bb\)? What about the patterns \(aaa\) and \(bbb\)? Can they ever occur in the word above? The pattern \(aa\) and \(bb\) are examples of “squares” and the patterns \(aaa\) and \(bbb\) are examples of “cubes.” In general a square pattern in a word is a sequence of symbols \(xx\) where \(x\) itself may be some complex string. Thus, \(abaab\) is an example of a square.

**Theorem:**

Any infinite word that uses only two symbols cannot have a square pattern.

Here is a proof using 0 and 1 as the symbols of the alphabet, since clearly it does not matter what two distinct symbols we use in our two-symbol alphabet.

Suppose the infinite word \(W\) begins with the symbol 0. If the next symbol is a zero we have the square 00, so the second symbol in \(W\) must be a 1. So far \(W\) looks like 01. Now what can be the next symbol of \(W\)? If it is a 1, we would have the square 11 in our word, so the next symbol would have to be a 0. Now \(W\) must begin 010. And now we are stuck! If the next symbol is a 0 we have the square 00, while if the next symbol is a 1 we have 011 using the familiar exponential notation to mean 011. A very similar argument can be made if our word starts with 1 instead of zero.

Note that this theorem applied not only to the particular two-symbol
Fibonacci word but also to ANY infinite word constructed from 2 symbols.

So an infinite string in a two-letter alphabet must have squares. But what about cubes? A quick look at a part of the infinite Fibonacci string above may not show any cube but, in fact, there are cubes in this string. Can you find a copy of (baaba)3 in the string (***)? It turns out there is one!

Here is a short list of interesting “facts” about the infinite Fibonacci word:

a. The substrings 11 and 000 never appear.

While the Fibonacci sequence must contain squares, one can still investigate which squares it cannot have and which ones do appear.

b. If W is a subword of the infinite Fibonacci word and one writes down the symbols of W in reverse order, then this new word will also be a subword of the infinite Fibonacci word.

c. If W is a subword of the infinite Fibonacci word, W appears infinitely often.

Perhaps you find the issue of finding particular patterns of strings in a word “silly” or perhaps you find questions of this kind really intriguing. However, questions of this type do indeed have applications outside of mathematics. With the insight we have obtained about genetics from the Crick-Watson model for inheritance, we know that genetic information is stored in “strings” of DNA which can be coded using the 4-letter alphabet, A, T, G and C. These letters stand for the nucleotides adenine (A) and thymine (T), guanine (G) and cytosine (C) (uracil (U) replaces thymine in RNA). A always pairs with T in DNA, A pairs with U in RNA, and G pairs with C in both DNA and RNA.

So there may be a sequence of letters involving A, T, C, and G that has been shown to have a certain function in a mouse, and now one wants to see if there is an equivalent sequence of letters somewhere in the human genome. This leads to interest in questions about the fastest algorithm which will find a particular string in another string that involves the same alphabet. It raises interest in palindromes, strings that read the same way in both directions. Examples of such words in English are racecar and civic. The way that DNA is shuffled in the reproductive process has lead to interest in DNA strings that are either palindromes or near palindromes, with a definitional twist. You will recall that DNA has a structure of a double helix consisting of two strands. The patterns of letters in one strand codes with A, T, C, G but the pattern in the other strand is always determined by the fact that A is always paired with T in the other strand, and similarly with C and G. Thus if one has the word AATCGT on one strand, the other strand will have TTAGCA. So biologists are interested when one has a “palindromic” string, for example, AAGGTATACC, where one strand, and similarly with C and G. Thus if one has the word AATCGT on one strand, the other strand will have TTAGCA. So biologists are interested when one has a “palindromic” string, for example, AAGGTATACC, where its complement TTCCATATGGAA, when read in reverse order, gives the original string.

Algorithms that involve the manipulation of strings also come into play in software that is used to detect plagiarism—the using of someone else’s words as if they were your own. Software can be designed that looks for “exotic” phrases and then searches for this exact phrase in documents that have been put onto the World Wide Web.

While many branches of mathematics have their natural roots in questions arising in science (physics, chemistry, engineering), in large countries such as Germany, France, England, Italy or Spain, the earliest work on the branch of theoretical mathematics now dubbed the combinatorics of words started in Norway and was developed in the early 20th century. Furthermore, relatively little work in this area was done until much later—the last 40 years.

**Combinatorics on Words**

What today is known as the Combinatorics of Words is tied to an important sequence in the spirit of the infinite Fibonacci word we have looked at already, which is now known as the Thue-Morse sequence or, sometimes, as the Prouhet-Thue-Morse sequence. Before saying more about this sequence itself, let me recount some of the history.

Axel Thue was a Norwegian mathematician who lived from 1863 to 1922. He was the only doctoral student of Elling Holst who, in turn, was a student of the famous Norwegian mathematician M. Sophus Lie. Axel Thue had only one student, the logician Thoralf Skolem, who achieved much more fame than his thesis advisor. Thue was primarily a number theorist. His most famous work was on the approximation of real numbers using rational numbers, but he became interested in the behavior of strings, seemingly because of his interest in the theory of groups (abstract algebra). In all, Thue wrote several papers about what has come to be called the combinatorics on words. In reviewing Thue’s collected works, published in 1977, Wolfgang Schmidt writes: “Thue was far ahead of his time.” This is reflected in the fact that much of what he did had no immediate effect on research about combinatorics of words. Partly because his work was published in “offbeat” journals, it was not picked up immediately; it was only noticed much later how important his ideas were.

The other name associated with what is today known as the Thue-Morse sequence is that of Marston Morse. Morse was an American mathematician who lived from 1892 to 1977. Morse independently of Thue discovered the Thue-Morse sequence in the context of work in what has come to be called symbolic dynamics, which is part of the broader subject of dynamical systems that arises in topology and analysis. Loosely
speaking, the idea is to study what happens when a function is applied to an input value, and then one applies the function again to the output of what one got from applying the function the first time, generating what are called “orbits.”

**Thue-Morse Sequence**

Before dealing with the truly extraordinary Thue-Morse sequence, let me say a bit more about how better known issues are related to some topics that come up in combinatorics of words. Strings or words are built up from symbols in an alphabet. For example, with an alphabet of two symbols, say a and b, we can look at all the strings of length 4. We can enumerate all of the strings but is there a way of counting how many strings there are? Since each position in the word being formed of length 4 can be one of two choices, we see that the first element of the string can be one of two symbols and that after this choice we can get the second element in two different ways. Using what is often referred to as the “fundamental principle of counting” we can see that there are 16 or 2⁴ different strings. Here is a sample:

- aaaa
- baaa
- bbba
- aabb
- bbaa
- bbbb

Notice that the kinds of questions we are interested in don’t govern what choice of symbols we would use to represent the strings.

Thus the strings above could just as well have been written:

- 0000
- 1000
- 0001
- 0011
- 1100
- 1111

However, one advantage of using binary strings with the symbols 0 and 1 is that we can choose, if we want, to interpret the strings more easily as numbers. So in binary, 1100 represents the number 12 in base 10. Binary sequences also find applications in “codes” of different kinds. When one sees sequences involving zeros and ones, one may have a richer collection of associations than would be the case when the alphabet of two letters is a, b or +1, −1 or X, Y.

One can use combinatorics (fundamental principle of counting) to count how many words there are with a k-letter alphabet that have length r (k at least r) and no repeated letters and ones where repeats occur.

One has to be careful what the “rules” are in discussing words. Is one interested in words only of a fixed length? Is one interested in a sequence of words? Is one interested in a string that consists of a finite number of consecutive positions in an infinite word?

To try to put your head around the distinctions here we will look at a remarkable collection of ideas that started as a contribution to theoretical mathematics over a hundred years ago. It got much less attention than it deserved because it was published in a relatively obscure journal. The notion was rediscovered later by the mathematician Marston Morse, again in a setting that was highly theoretical but quite a different setting from Axel Thue’s work.

Let us try to be relatively specific but adopt a general framework for looking at “words.” We will assume we are dealing with a finite alphabet of symbols. We will refer to sequences of letters drawn from the alphabet as words. Sometimes one is interested in finite-length words, sometimes one-way infinite words, sometimes two-way infinite words, and sometimes words whose letters are arranged in a “circuit”—cyclic words.

Here are some examples of words drawn from the three-letter alphabet a, b, c:

- abcabca (finite length word)
- abacabcbacbbacabbac... (one-way infinite word)
- ....abcababca... (two-way infinite word)

Henceforth, we will only consider finite or one-way infinite words. Now - at last, the Thue-Morse sequence. We have already noted above that any infinite word made up of zeros and ones must contain a square. A natural question is to see what kinds of patterns can be avoided in an infinite word of zeros and ones. Before you see the Thue-Morse sequence, consider the following result.

**Theorem**

(Thue-Morse) There exists a sequence of zeros and ones with the property that it has no cubes (e.g. strings of the form xxx).

We saw previously that the Fibonacci word does have cubes, and we will see below that the Thue-Morse obeys much stronger conditions than being cube-free. The fact that the Thue-Morse sequence is fundamental also attested to by the wide variety of settings where it arises and the many different approaches to constructing it.

We will use TMₙ to denote the nth term of the Thue-Morse sequence, where n takes on the values n = 0, 1, 2, 3, ... to denote the terms in the Thue-Morse sequence which I will sometimes refer to as the T-M sequence.

To find the nth term of the T-M sequence, convert the index n to its binary representation. If this representation has an even number of 1’s, then TMₙ is 0. Otherwise it is 1. Thus, the nth entry in the sequence depends on the number of ones in the binary representation of the index n. When the number of 1’s is even (e.g. 0, 2, 4, etc.), we set the term of the sequence to 0; otherwise we set the nth entry to 1.
To give you the idea, shown in Table 1 are the decimal integers from 0 to 16, their binary representations, and the “parity” of the number of 1’s in the binary representation of each index \( n \). When the number of 1’s in the binary expression for \( n \) is even, the entry in the third column is 0. Otherwise it is one. Another way of saying how one computes the entry in the third column is that it is the remainder when the number of 1’s in column 2 is divided by 2. This remainder will always be either 0 or 1. This remainder gets recorded in column 3.

<table>
<thead>
<tr>
<th>( n ) in decimal</th>
<th>( n ) in binary</th>
<th>Number of 1’s mod 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>101</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>110</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>111</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>1000</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>1001</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>1010</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>1011</td>
<td>1</td>
</tr>
<tr>
<td>12</td>
<td>1100</td>
<td>0</td>
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<tr>
<td>13</td>
<td>1101</td>
<td>1</td>
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<tr>
<td>14</td>
<td>1110</td>
<td>1</td>
</tr>
<tr>
<td>15</td>
<td>1111</td>
<td>0</td>
</tr>
<tr>
<td>16</td>
<td>10000</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1

3 one’s, we know that the \( TM_{50} = 1 \).

Are there other ways of generating T-M?

Here is a recursive approach to defining the entries of the sequence!

\[
TM_0 = 0
\]

(a) \( TM_{2n} = TM_n \) and (b) \( TM_{2n+1} \) is not equal to \( TM_n \)

What is going on here? When a number is doubled, what happens to its binary representation? The answer is that a zero gets added at the end, which means that the number of 1’s in the representation stays the same.

So \( TM_0 = 0 \) and using (b) with \( n = 0 \) we get that \( TM_1 = 1 \). Now \( n = 1 \), we use (a) to conclude that \( TM_2 \) is the same as \( TM_1 \) or 1, while using (b) with \( n = 1 \) we see that \( TM_3 \) should be different from \( TM_2 \) and, thus, be 0. Continuing on in this manner we get the same values as we did previously.

If you think through the meaning of the way that the T-M is generated above, there is another nifty way of generating the T-M:

From the sequence listed from \( TM_0 \) to \( TM_2 \) we will see how at each stage we can double the number of entries we have in the ‘infinite word.’ Write down the result from the sequence listed from \( TM_0 \) to \( TM_2 \) and now write down this string next to the original but with the roles of 0 and 1 interchanged. Here is what you get when you carry this out:

0
01
0110
01101001
01101001100110110011011001101001

So here is what happens:

0
01
0110
01101001
01101001100110110011011001101001

As you can see, this is our “old friend,” the T-M sequence.

For those familiar with function notation here is how one expresses this:

\( tm(0) = 01; tm(1) = 10 \). T-M can be written \( tm^{\infty}(0) \). This last means \( tm(tm(tm(...(tm(0)...))) \)). We compute the result of applying the function to get an answer starting with the string 0, and repeat applying the “substitution map” over and over again.

So, what is it that is interesting about the Thue-Morse sequence? We have already seen that with only an alphabet of two symbols we can’t construct long words that don’t have squares (e.g. strings like \( xx \)). This raises a variety of questions of both theoretical and applied interest:

- **a.** What are the fewest symbols that an alphabet can have and yet admit square-free arbitrarily large words?

We will see that Axel Thue was able to construct a sequence of this kind, employing ideas related to the T-M sequence, using only an alphabet of 3 symbols—a remarkable result.

- **b.** To what extent does the T-M sequence avoid having overlaps? An overlap in an infinite (one-way) string is a substring which looks like \( aZaZa \) where \( a \) is any single letter in the alphabet and \( Z \) is any string, possibly of length 0. When \( Z \) has
length 0, then the pattern aZaZa becomes a single letter cubed (i.e., aaa). If there is no pattern of the kind aZaZa, then the (infinite) word is said to be overlap-free.

Why are overlaps of interest? Years after Thue’s work biologists are trying to understand the meaning of strings of DNA in different species. Each species has a collection of genes which contain the genetic information for the species. For a long time the DNA outside the stretches which “code” information was thought to be “junk.” This referred to DNA that was there but did not have any part in inheritance issues. We now know the situation is much more complex, and that DNA outside of sections that are genes may have “regulatory” or other information that governs the way(s) genes are “expressed.” Very often geneticists are interested in “tandem repeats,” although, loosely speaking these are stretches of DNA that are “almost powers” and, hence, are “modeled” by overlaps. Locating such sections in DNA is valuable in determining a particular person’s parentage and other aspects of genome determination.

**Theorem**
The Thue-Morse infinite word (which uses only two symbols) is overlap-free.

Here is a way to construct a word that is square-free that uses only three symbols and which is based, you guessed it, on the Thue-Morse sequence.

We will use only the symbols 0, 1, 2 in the word we construct, and we will denote the sequence $S_F^n$ (for square-free) where $n = 1, 2, 3, \ldots$.

$S_F^n$ will denote the number of 1’s between the $n$th and $(n+1)$st zeros in the T-M sequence. Here is a stretch of the T-M sequence:

011010011001011001101001

$S_F^1$ is 2, since first 1 is in position 0, and the second 0 is in position 3, so there are two one’s between these positions.

$S_F^2$ is 1 since the second 0 is in position 3 and the third 0 is in position 5 so there is one one between them.

$S_F^3 = 0$

$S_F^4 = 2$

$S_F^5 = 0$

$S_F^6 = 1$

$S_F^7 = 2$

which gives the initial section of the $S_F$ infinite word:

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It is easy to see that the distance between two ones in the T-M sequence can only be 0, 1, or 2. Otherwise we would have as many as three ones in a row, the word 111, which would mean that the T-M sequence contained a cube, which (though we did not include the proof) cannot occur. Were this sequence to have a square, it can be shown that the T-M sequence would have to have an overlap, which cannot be the case. One can also find morphism approaches to constructing square-free words with an alphabet of only size three.

The Thue-Morse, though seemingly having a primarily combinatorial and algebraic quality, has also inspired geometrical work and artistic work. Tilings, freezes, knots, and fractals all have been investigated inspired by the Thue-Morse sequence. A sample appears at the start, due to Mark Dow who works at the brain development laboratory at the University of Oregon.

In recent years there has been a tremendous explosion of results in the combinatorics of words, connections being found to matrix theory, automata theory, graph theory, etc. One origin of this work is the theoretical work of Axel Thue. But progress in the field is nurtured by mathematical intellectual curiosity and the applications that the field of combinatorics on words has made possible.

**Acknowledgment**
This article is a modified version of a web-based article that I wrote and appeared as an American Mathematical Society Feature Column in September, 2013.

**Bibliography**


Lothaire, M., Combinatorics on Words, Addison-Wesley, 1983.


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Joseph Malkevitch
Department of Mathematics and Computer Studies
York College (CUNY)
Jamaica, New York 11451
Although it feels as through at the last minute we were asked to put together a mathematical modeling summer program for a school in Carmel, CA, it was worth the effort. It was a nice experience to think about and to create such a summer program. This article suggests a straw man to create such a program.

First, it would be nice to have a full week or at least 2 to 2 ½ days for a workshop for mathematics teachers in our schools. It would be nice to have sponsorship so that the teachers could get a small stipend in addition to the workshop materials. Our coverage was similar to the following:

- Present a modeling problem of interest in the area or to society
- Present mathematical tools & examples (standards) and technology
- Have participants work on model
- Discuss and share results

We want to get the teachers excited, so they can get their students excited about mathematical modeling.

The following is our schedule for a two-day workshop.

### Day 1

**Introduction:** Mathematical Modeling Process

**Session 1: DDS and Technology**

- Inheritance Model
- Financial Models
- Systems of Systems
- Predator-Prey and SIR Models

**Session 2: Data Analysis**

Activity one: Sports

Activity two: Present data from 1880-2004 (Table 1 below). Is there evidence for global warming? Let the participants work on this question.

Activity three: Global warming. Why start with the 1880s? (2nd Industrial Revolution around 1850, machines, railroad, etc.)

**Session 3: Models with Data, & Regression**

- Spring-Mass experiment (Hooke’s Law)
- Distance, velocity, and acceleration models: breaking distance or telemetry data. Catapult experiment: Teachers use an actual catapult to collect data and build a model (Figure 1).

**Session 4: Modeling with Linear Systems of Equations**

- Chemical Balancing as a system of linear equations using matrices.

**Session 5: Linear Optimization and Game Theory**

- HiMCM Forest fire problem
- Katrina Aid and Rescue (Linear Programming—2 variable)

**Table 1**

<table>
<thead>
<tr>
<th>Central England Average Temperatures (°C)</th>
<th>JAN</th>
<th>FEB</th>
<th>MAR</th>
<th>APR</th>
<th>MAY</th>
<th>JUN</th>
<th>JUL</th>
<th>AUG</th>
<th>SEP</th>
<th>OCT</th>
<th>NOV</th>
<th>DEC</th>
<th>Annual</th>
</tr>
</thead>
<tbody>
<tr>
<td>1961 to 1990</td>
<td>3.8</td>
<td>3.8</td>
<td>5.7</td>
<td>7.9</td>
<td>11.2</td>
<td>14.2</td>
<td>16.1</td>
<td>15.8</td>
<td>13.6</td>
<td>10.6</td>
<td>6.6</td>
<td>4.7</td>
<td>9.5</td>
</tr>
<tr>
<td>1880 to 2004</td>
<td>3.8</td>
<td>4.1</td>
<td>5.7</td>
<td>8.0</td>
<td>11.3</td>
<td>14.2</td>
<td>16.1</td>
<td>15.7</td>
<td>13.5</td>
<td>10.0</td>
<td>6.5</td>
<td>4.5</td>
<td>9.5</td>
</tr>
<tr>
<td>1659 to 2005</td>
<td>3.2</td>
<td>3.8</td>
<td>5.3</td>
<td>7.9</td>
<td>11.2</td>
<td>14.3</td>
<td>16.0</td>
<td>15.6</td>
<td>13.3</td>
<td>9.7</td>
<td>6.0</td>
<td>4.1</td>
<td>9.2</td>
</tr>
</tbody>
</table>
Session 6: Transformed models: ln-ln Models

Terror Bird Problem
Water tank problem (from MCM)

Session 7: Simulation Models

Search and Find – Previous HiMCM Problem
Bank queues – Previous HiMCM Problem

Session 8: Discuss previous HiMCM problems not covered of interest such as:

Water, Water Everywhere

Fresh water is the limiting constraint for development in much of the United States. Devise an effective, feasible, and cost-efficient national water strategy for 2010 to meet the projected needs of the United States in 2025. In particular, address storage and movement, de-salinization, and conservation as some of the possible components of your strategy. Consider economic, physical, cultural, and environmental effects. Provide a position paper for the United States Congress outlining your approach, its costs, and why it is the best choice for the nation.

Tsunami (“Wipe Out!”)

Recent events have reminded us about the devastating effects of distant or underwater earthquakes. Build a model that compares the devastation of various-sized earthquakes and their resulting Tsunamis on the following cities: San Francisco, CA; Hilo, HI; New Orleans, LA; Charleston, SC; New York, NY; Boston, MA; and any city of your choice. Prepare an article for the local newspaper that explains what you discovered in your model about one of these cities.

Session 9: Optional Topics

Careers in applied mathematics
Modeling process
Mathematics topics, as desired
Additional previous HiMCM problems

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William P. Fox (ed.) is a professor at the Naval Postgraduate School where he has been since 2006. Prior to that he was professor & chair of mathematics at Francis Marion University and he was a professor at the United States Military Academy for about 12 years. His major scholarly interests include: applied mathematics, mathematical models, and optimization as well as mathematical modeling contests where he serves as the HiMCM Contest Director and MCM Contest Director. HiMCM Notes concerns insights that focus on the High School Mathematical Contest in Modeling (HiMCM). The purpose is to provide teachers, advisors, schools, and students with information, strategies, and examples to establish and prepare teams to compete in the HiMCM.

For more information contact: William P. Fox, Department of Defense Analysis, Naval Postgraduate School, Monterey, CA 93943
E-mail: wpfox@nps.edu
This Pull-Out provides real-world settings that guide students through matrix addition and subtraction, and scalar and matrix multiplication. In Activity 1, students store prices of pizzas, salads, and soft drinks from three pizza houses into a matrix. Matrix addition is used to revise the prices to reflect the cost of additional toppings and choices of salad dressings. After organizing information on coupons into a matrix, matrix subtraction is used to apply the coupons and reduce the costs. At the end of Activity 1, students learn how to store matrices in TI-84 graphing calculators. Then they use their calculators to determine sums and differences of two matrices. (Instructions on using Excel are included in the Lesson Notes.) In Activity 2, students use scalar multiplication to compare the prices of ordering $k$ pizzas and $k$ salads from each of the pizza houses. Matrix multiplication is introduced in three steps: (1) multiplying a row matrix and a column matrix, (2) multiplying a row matrix and a column matrix, and finally, (3) multiplying a multi-row matrix and a multi-column matrix. Then matrix multiplication is used to compare three possible options for the purchases of pizzas and salads at the three pizza houses. At the end of Activity 2, students use their calculators to investigate whether the Associative and Commutative Laws for Addition and Multiplication, which students have learned in their algebra classes, also hold for matrices. Activity 3 focuses on use of matrices to investigate a population growth model called the Leslie model. Students use their calculators (or Excel) to approximate the age distribution of a population and the size of the total population into the future.

The activities in this Pull-Out address standard N-VM-(6-9) from the Common Core State Standards for High School Mathematics—Perform operations on matrices and use matrices in applications. Here are the details of Items 6 – 9:

6. Use matrices to represent and manipulate data.
7. Multiply matrices by scalars to produce new matrices.
8. Add, subtract, and multiply matrices of appropriate dimensions.
9. Understand that, unlike multiplication of numbers, matrix multiplication for square matrices in not a commutative operation but still satisfies the associative and distributive properties.

The activities in this Pull-Out are adapted from Discrete Mathematics: Modeling Our World by Nancy Crisler and Gary Froelich. More information on the newly released edition of this text is available at www.COMAP.com.

Dr. Marsha Davis is a Professor at Eastern Connecticut State University in Willimantic, CT. She can be reached at davisma@easternct.edu or at the Department of Mathematics and Computer Science, Eastern Connecticut State University, 83 Windham Street, Willimantic, Ct, 06226.
Activity 1: Pizzas, Salads, and Matrices

Activity 1 introduces the use of a matrix to store a rectangular array of numbers. Make sure that students are comfortable with identifying the dimension of a matrix. When referring to an $m \times n$ matrix, $m$ is the number of rows and $n$ is the number of columns. In the case where $m = n$, the matrix is called a square matrix. Students should also be comfortable with the convention of notating a specific entry in a matrix using a double subscript. For example, $A_{23}$ represents the entry in row 2 and column 3 of matrix $A$.

Two matrices of the same dimension can be added or subtracted. Instead of describing the addition or subtraction process in words, as was done in Activity 1, you could specify each element as follows: Given two matrices $A$ and $B$ of the same dimension, $C = A + B$, and $D = A – B$, then

- $C_{ij} = A_{ij} + B_{ij}$
- $D_{ij} = A_{ij} - B_{ij}$

At the end of the activity in Exercises 8 – 11, students enter matrices into TI-84 graphing calculators and use their calculators to perform matrix addition and subtraction. If your students are using a different calculator that has matrix capabilities, adapt the TI-84 instructions for use with their calculators. If you want to give students experience with Excel, the Excel instructions are given below.

Excel Instructions.

The nice thing about using Excel is that you can include the labels for the rows and columns of your matrix. So, for example, you could enter matrices $A$ and $B$ from Exercise 8 into an Excel spreadsheet as shown in Figure 1. For $A$ we have added row and column labels. We have also specified the matrix names and put a box around the matrices for clarity, but there is no need to do so.

To add two matrices, first select a rectangle of empty cells with the same dimension as the result. In this case, the result of adding two $3 \times 2$ matrices is a $3 \times 2$ matrix. (See Figure 2.)

Enter the formula: $= A + B$ and then simultaneously press CTRL SHIFT ENTER. To subtract two matrices, just replace the $+$ in the formula with $–$.

Activity 2: Matrix Multiplication Served with Salad and Pizza

The mathematics in Activity 2 focuses on matrix multiplication. Students are eased into matrix multiplication gradually.

1. First, they multiply a row matrix and a column matrix. The number of columns in the row matrix must match the number of rows in the column matrix.
2. Then they adapt the process used in (1) to multiply a row matrix and a multi-column matrix.
3. Finally, they adapt the process used in (2) to multiply a multi-row matrix and a multi-column matrix.

You may want to review matrix multiplication at some point to make sure that students understand how it works. For example, consider the matrices below.

We can perform the matrix multiplication to find $P = C \times D$ since the number of columns of matrix $C$ matches the number of rows of matrix $D$. To find the entry $P_{21}$, we multiply corresponding entries in row 2 of $C$ and column 1 of $D$ as shown below.

Hence, $P_{21} = C_{21}D_{11} + C_{22}D_{21} + C_{23}D_{31} + C_{24}D_{41} = 3(3) + 6(2) + 7(1) + 9(0) = 28$. The double subscripts can be confusing to students. However, the multiplication process can be understood even if the notation used to express that process is not clearly understood.
At the end of Activity 2, students enter three $3 \times 3$ matrices into their calculators. Since you can only add or multiply two matrices at a time, the question to ask is: Does it matter which two matrices you perform the operation on first? This question leads to the Associative Properties of Addition and Multiplication.

**TI-84 Instructions (Adapt for Other Calculators)**

One word of warning, a TI-84 will give an error message if a student enters the following: ([A][B])[C]. Instead, students must hit the multiplication key in order to multiply: ([A]×[B])×[C]. However, if a student just enters [A][B][C] with no parentheses, the calculator will have no problem computing the matrix multiplication.

In Activity 3, students will need to raise a square matrix to a power. (You might ask students why only square matrices can be raised to a power.) On a TI-84, here’s how to raise a square matrix to a power, say 3: After selecting the matrix, press $^3$, and then ENTER.

If your students are using a calculator different from the TI-84, you will need to adapt the TI-84 calculator instructions.

**Excel Instructions**

Enter the matrices and name them as shown in the Lesson Notes for Activity 1. To multiply two matrices $A \times B$, select a rectangle of empty cells of the appropriate dimension for the product. Enter the formula =A*B and then simultaneously press CTRL SHIFT ENTER.

To raise a matrix $A$ to a power, say 3, select a rectangle of empty cells of the appropriate dimension for the result. Enter the formula = $A^3$ and then simultaneously press CTRL SHIFT ENTER.

**Activity 3: Population Growth: The Leslie Model**

For Activity 3, students need to have access to technology for the matrix calculations. In Exercises 8 and 9 students will use trial and error to determine the number of cycles it will take for a population (either or rats or deer) to exceed the carrying capacity of the natural habitat in which these animals are living. After exceeding the carrying capacity, the animals will begin to die off due to overcrowding.

**TI-84 Instructions**

In Exercises 8 or 9, students can start with a guess, say 10 cycles, and then calculate the total female population by computing $P_0L^{10}$. To compute the total population for a different number of cycles, say 20, they need only press 2nd ENTRY, change the 10 to 20 and then press ENTER. This shortcut will speed up the calculations.

**Modeling Project**

If students google or scholar google “leslie matrix model,” they will have a wide selection of sites to investigate.
**Answers to Activity 1:**
**Pizzas, Salads, and Matrices**

1. **a**  
   \( A_{31} \) is the entry in row 3 and column 1 of matrix A. The value of that entry is $3.69, which is the price of salad at Vin’s.

2. **b**  
   \( A_{21} = \$1.09; A_{12} = \$10.86; A_{32} = \$3.69. \)

3. **c**  
   \( A_{31} \) is the entry in row 2 and column 1, the price of drinks at Vin’s; \( A_{12} \) is the entry in row 1 and column 2, the price of pizza at Toni’s; \( A_{32} \) is the entry in row 3 and column 2, the price of salad at Toni’s.

4. **b**  
   \( B_{12} = \$10.86; C_{12} = \$1.10; D_{12} = \$10.86 + \$1.10 = \$11.96. \)

5. **b**  
   \( E_{12} = \$2.00; F_{12} = \$4.14 + \$3.35 = \$7.49. \)

6. **a**  
   You can’t add a 2 \( \times \) 3 matrix and a 3 \( \times \) 2 matrix because they do not have the same dimension.

   **b**  
   \[ \begin{bmatrix} 9 & -4 \\ -4 & 7 \end{bmatrix} \]

   **c**  
   \[ \begin{bmatrix} 0 & -5 \\ 6 & 0 \\ 3 & 4 \\ 0 & -2 \end{bmatrix} \]

**Activity Answers HiMAP Pull-Out Section: Fall/Winter 2014**

8. **d**  
   You can’t subtract a 3 \( \times \) 1 matrix from a 1 \( \times \) 3 matrix because they do not have the same dimension.

9. **a**  
   They are all 0.

10. **b**  
   Yes. For example, \( R_{32} = 1,105 \) and \( R_{32} = 1,015. \)

11. **c**  
   Sample answer: \[ \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 1 & 3 & -1 \end{bmatrix} \]

**Sample answer:** Suppose \( A \) is a 3 \( \times \) 4 matrix. In order for \( A \) to be symmetric, entry \( A_{14} = A_{41}. \)

Although matrix \( A \) has an entry in row 1 and column 4, since matrix \( A \) does not have a row 4 it does not have an entry in row 4 and column 1.

We entered the data from 2011 and 2012 into two 3 \( \times \) 6 matrices that we named \( A \) and \( B, \) respectively. To find the changes from 2011 to 2012, we determined \( B - A. \) Negative values indicate that the value of the statistic when down. For example, compared to the 2011 season, M. Kemp had a very bad 2012 season. He had 16 fewer home runs (HR) in 2012 compared to 2011.

<table>
<thead>
<tr>
<th></th>
<th>AB</th>
<th>R</th>
<th>H</th>
<th>HR</th>
<th>RBI</th>
<th>Avg</th>
</tr>
</thead>
<tbody>
<tr>
<td>J. Reyes</td>
<td>105</td>
<td>-15</td>
<td>3</td>
<td>4</td>
<td>13</td>
<td>-0.05</td>
</tr>
<tr>
<td>R. Braun</td>
<td>35</td>
<td>-1</td>
<td>4</td>
<td>8</td>
<td>1</td>
<td>-0.013</td>
</tr>
<tr>
<td>M. Kemp</td>
<td>-199</td>
<td>-41</td>
<td>-73</td>
<td>-16</td>
<td>-57</td>
<td>-0.021</td>
</tr>
</tbody>
</table>

**Jackets:** Shirts  Pants

**Outlet 1:**
- Jackets: 355
- Shirts: 361
- Pants: 282

**Outlet 2:**
- Jackets: 196
- Shirts: 320
- Pants: 239

**Outlet 3:**
- Jackets: 221
- Shirts: 469
- Pants: 499

**Outlet 1:**
- Jackets: 109
- Shirts: 163
- Pants: 120

**Outlet 2:**
- Jackets: 78
- Shirts: 150
- Pants: 113

**Outlet 3:**
- Jackets: 102
- Shirts: 142
- Pants: 153
Activity Answers

Answers to Activity 2: Matrix Multiplication
Served with Salad and Pizza

1. 

\[
6D = 6 \begin{bmatrix}
11.25 & 11.96 & 11.90 \\
3.69 & 4.14 & 4.35 \\
\end{bmatrix} = \begin{bmatrix}
67.50 & 71.76 & 71.40 \\
22.14 & 24.84 & 26.10 \\
\end{bmatrix}
\]

2. (a) 

\[
Q = \begin{bmatrix}
\text{Chips} & \text{Candy} & \text{Gum} & \text{Drink} \\
1 & 2 & 2 & 1 \\
\end{bmatrix}; \\
P = \begin{bmatrix}
\text{Chips} & \text{Candy} & \text{Gum} & \text{Drink} \\
30 & 35 & 25 & 75 \\
\end{bmatrix}
\]

\[
QP = \begin{bmatrix}
1 & 2 & 2 & 1 \\
\end{bmatrix} \begin{bmatrix}
30 & 35 & 25 & 75 \\
\end{bmatrix} = \begin{bmatrix}
1(30) + 2(35) + 2(25) + 1(75) = 225 \\
\end{bmatrix}
\]

The total price of Terri’s purchases is 222 cents or $2.25.

3. (a) 

It is not possible to multiply 1 \times 5 row matrix times a 4 \times 1 column matrix. When you try to multiply the 5th entry from the row matrix with the 5th entry in the column matrix, there is no 5th entry.

(b) 

Treat each column in \( D \) as a column matrix. For each column, multiply its entries by the corresponding row entry and then sum the products. This process should result in 3 sums, one for each column in \( D \). The result was a 1 \times 3 matrix.

4. (a) 

Yes. The entries in \( A \) can multiply the corresponding row entries in each of \( B \)'s 2 columns. The dimension of \( AB \) is 1 \times 2.

(b) 

No. The number of columns in \( A \), which has 4 columns, must match the number of rows in \( B \), which only has 3 rows.

5. 

\[
VW = \begin{bmatrix}
29 & 72 \\
\end{bmatrix}
\]

Entry \( E_{23} \) represents the cost of Option 2 at Sal’s; entry \( E_{32} \) represents the cost of Option 3 at Toni’s.

6. 

\( NM \) is defined because there are 2 columns in \( N \), which matches the number of rows in \( M \).

7. 

\( 4 \times 3 \)

(b) 

\[
P_{23} = 1(1) + 1(1) + 4(1) + 2(3) + 0(1) = 12.
\]

(c) 

\[
P_{41} = 2(2) + 8(3) + 3(1) + 1(2) + 6(3) = 51.
\]

9. (a) 

\[
A = \begin{bmatrix}
12 & 32 & -4 \\
8 & 0 & 16 \\
\end{bmatrix}
\]

9. (b) 

\[
AB = \begin{bmatrix}
7 & 5.5 & 7 \\
4 & 6 & 2 \\
\end{bmatrix}
\]

(c) 

\( BA \) is not defined. The number of columns in \( B \), 3, does not match the number of rows in \( A \), 2.

(d) 

\[
CA = \begin{bmatrix}
7 & 8 & 7 \\
24 & 32 & 20 \\
\end{bmatrix}
\]

(e) 

Since \( A \) and \( C \) do not have the same dimension, you cannot add them.

Method #1: You could begin by finding the tax matrix by multiplying matrix \( E \) (Matrix 13) by the scalar 0.0625. Then add this tax matrix to matrix \( E \).

Tax Matrix: We decided to always round entries containing partial cents up – thinking that Massachusetts would charge, for example, a tax of $1.25 if the computed tax was only $1.241. (The state would want the additional penny in tax.)

Method #2. You could simply multiply matrix \( E \) by the scalar 1.0625.

Matrix \( Q \):

\[
\begin{bmatrix}
3.51 & 3.77 & 3.80 \\
3.74 & 4.03 & 4.07 \\
4.21 & 4.52 & 4.54 \\
\end{bmatrix}
\]

Matrix \( C \):

\[
\begin{bmatrix}
450 & 40 & 50 \\
570 & 48 & 90 \\
500 & 45 & 25 \\
300 & 30 & 0 \\
400 & 22 & 50 \\
\end{bmatrix}
\]

The dimensions of matrices \( Q \) and \( C \) are \( 2 \times 5 \) and \( 5 \times 3 \), respectively.

The dimension of the product \( QC \) is \( 2 \times 3 \).
Activity Answers

11 e The dimension of C can be described as Foods by Contents, and the dimension of Q times C can be described as Persons by Contents.

\[
\frac{Q}{\text{Persons by Food}} \times \frac{C}{\text{Foods by Contents}} = \frac{P}{\text{Persons by Contents}}
\]

Dimensions of Product

11 f \[ R = \begin{bmatrix} \text{Cal} & \text{Fat} & \text{Chol} \\ \text{Max} & 1,270 & 100 & 140 \\ & 1,350 & 107 & 125 \end{bmatrix} \]

11 g \[ R_{12} \text{ represents the total grams of fat in Rosa’s food; } R_{23} \text{ represents the total number of calories in Max’s food; } R_{31} \text{ represents the mg of cholesterol in Max’s food.} \]

12 \[ AB = \begin{bmatrix} 10 & 9 & 14 \\ -2 & 6 & -3 \\ 1 & 6 & 1 \end{bmatrix} \]

13 a \[ A + B = \begin{bmatrix} 1 & 7 & 3 \\ -1 & 1 & 7 \\ -1 & 4 & 3 \end{bmatrix} \]

13 b \[ B + A = \begin{bmatrix} 1 & 7 & 3 \\ -1 & 1 & 7 \\ -1 & 4 & 3 \end{bmatrix} \]

14 a \[ BA = \begin{bmatrix} 5 & 13 & 6 \\ -1 & 3 & 11 \\ 4 & 11 & 6 \end{bmatrix} \]

14 b \[ (A + B) + C = \begin{bmatrix} 5 & 13 & 6 \\ -1 & 3 & 11 \\ 4 & 11 & 6 \end{bmatrix} \]

15 a \[ A + (B + C) = \begin{bmatrix} 5 & 13 & 6 \\ -1 & 3 & 11 \\ 4 & 11 & 6 \end{bmatrix} \]

15 b \[ (A + B) + C = \begin{bmatrix} 5 & 13 & 6 \\ -1 & 3 & 11 \\ 4 & 11 & 6 \end{bmatrix} \]

16 a \[ A(BC) = \begin{bmatrix} 110 & 176 & 108 \\ 9 & 25 & 30 \end{bmatrix} \]

16 b \[ (AB)C = \begin{bmatrix} 110 & 176 & 108 \\ 9 & 25 & 30 \end{bmatrix} \]

17 a \[ A \times A = \begin{bmatrix} -18 & -3 & 24 \\ -7 & -7 & 9 \end{bmatrix} \]

17 b \[ A^2 = \begin{bmatrix} -18 & -3 & 24 \\ -7 & -7 & 9 \end{bmatrix} \]

17 c \[ B^3 = \begin{bmatrix} 44 & 51 & 36 \\ 51 & 47 & 51 \\ 36 & 51 & 23 \end{bmatrix} \]

Answers to Activity 3: Population Growth: The Leslie Model

1 a \[ \begin{bmatrix} 15 & 9 & 5 & 0 & 0 \\ 0.4 & 0.7 & 0.8 & 0.3 & 0.6 & 0 \end{bmatrix} \]

2 \[ \begin{bmatrix} 0 & 0.6 & (16.6) & 0 & 0 & 0 \end{bmatrix} \]

3 a \[ \begin{bmatrix} 16.6 & 9 & 81 & 11.7 & 4 & 0 \\ 0.7 & 0 & 0 & 0 & 18.97 \end{bmatrix} \]

3 b \[ \begin{bmatrix} 16.6 & 9 & 81 & 11.7 & 4 & 0 \\ 0.7 & 0 & 0 & 0 & 18.97 \end{bmatrix} \]

3 c Refute. Matrix multiplication does not satisfy the commutative property.

15 a \[ A + (B + C) = \begin{bmatrix} 5 & 13 & 6 \\ -1 & 3 & 11 \\ 4 & 11 & 6 \end{bmatrix} \]

15 b \[ (A + B) + C = \begin{bmatrix} 5 & 13 & 6 \\ -1 & 3 & 11 \\ 4 & 11 & 6 \end{bmatrix} \]

15 c Verify.

The total female rat population after 6 months (2 cycles) is 18.97 + 9.96 + 8.1 + 7.29 + 9.36 + 2.4 = 56.08, or approximately 56 rats (or 57 rats if we round up to the nearest integer).
**Activity Answers**

### 3 d

The number of new female babies is:

\[
\begin{bmatrix}
18.97 & 9.96 & 8.1 & 7.29 & 9.36 & 2.4
\end{bmatrix}
\begin{bmatrix}
0 \\
0.3 \\
0.8 \\
0.7 \\
0.4 \\
0
\end{bmatrix}
= \begin{bmatrix} 18.315 \end{bmatrix}.
\]

### 5 b

\[
P_t = P_0 L^t = \begin{bmatrix}
18.97 & 9.96 & 8.1 & 7.29 & 9.36 & 2.4
\end{bmatrix}
\begin{bmatrix}
0.6 & 0 & 0 & 0 & 0 & 0 \\
0.8 & 0 & 0 & 0.9 & 0 & 0 \\
0.7 & 0 & 0 & 0 & 0.8 & 0 \\
0.4 & 0 & 0 & 0 & 0 & 0.6 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
= \begin{bmatrix}
18.315 & 11.382 & 8.964 & 7.29 & 5.832 & 5.616
\end{bmatrix}.
\]

### 6

\[
P_t = P_0 L^t = \begin{bmatrix}
15 & 9 & 13 & 5 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0.6 & 0 & 0 & 0 & 0 & 0 \\
0.9 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
= \begin{bmatrix}
9
\end{bmatrix}.
\]

Total population = 21.03 + 12.28 + 10.90 + 9.46 + 7.01 + 4.27 = 64.95, or approximately 65 rats.

### 7 a

\[S = \begin{bmatrix}
1 \\
1 \\
1 \\
1 
\end{bmatrix}.
\]

### 5 a

\[
P_t = P_0 L^t = \begin{bmatrix}
15.9 & 13.5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\end{bmatrix}
= \begin{bmatrix}
\end{bmatrix}.
\]

### 6

\[
P_t = P_0 L^t = \begin{bmatrix}
15 & 9 & 13 & 5 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0.6 & 0 & 0 & 0 & 0 & 0 \\
0.8 & 0 & 0 & 0.9 & 0 & 0 \\
0.7 & 0 & 0 & 0 & 0.8 & 0 \\
0.4 & 0 & 0 & 0 & 0 & 0.6 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
= \begin{bmatrix}
18.315 & 11.382 & 8.964 & 7.29 & 5.832 & 5.616
\end{bmatrix}.
\]

Total population = 21.03 + 12.28 + 10.90 + 9.46 + 7.01 + 4.27 = 64.95, or approximately 65 rats.

### 7 b

\[P_{10}S = (P_0L^{10})S = 69.32, or approximately 70 rats.\]

(We rounded up to the whole rat.)

### 8 a

\[P_{41} = 245.7, which is not long enough. However,\]
\[P_{61} = 253.2; so, after 61 cycles, or 15.25 years the rat population will surpass 250 and begin to die off due to overcrowding.
\]

\[P_{41} = 249.6 and if you round up, the colony will have reached 250. So, after 41 cycles or 10.25 years the rat population will begin to die off due to overcrowding.
\]

### 9 a

The Leslie matrix \(L = \begin{bmatrix}
0.0 & 0.6 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.8 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
1.7 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
1.7 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.8 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.4 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0
\end{bmatrix}.
\]

### 9 b

\[P_S = \begin{bmatrix}
50 & 30 & 24 & 12 & 8
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix}
= 148
\]

### 9 c

\[(P_0L^5)S = 597.5.
\]

After 7 cycles, or 14 years, the herd size will be approximately 1,013 and will not have exceeded what the natural range can support. However, after 8 cycles or 16 years, the herd size will be approximately 1,328 female deer. This exceeds the capacity of the natural range for this animal and the deer will begin to die due to overcrowding.
Activity 1: Pizzas, Salads, and Matrices

At several points in the 1999 movie *The Matrix*, a rectangular array of numbers scrolled down the screen—a matrix. In the three activities in this Pull-Out, you will be using matrices to organize, manipulate, and display information. You can join other diverse groups such as baseball statisticians, business executives, and wildlife biologists who store, manage, manipulate, and calculate with matrices.

Suppose that you and a few of your friends are planning a pizza and video party. You decide to call several pizza houses to ask about prices for large single-top-ping pizzas, liter containers of soft drinks, and family-sized salads with house dressing. You organize the information gathered into the matrix shown below.

Matrix 1. Price matrix for food from four pizza houses.

Matrix 1 consists of three rows and four columns. We say that Matrix 1 has dimension $3 \times 4$ (read “3 by 4”). When writing this price matrix, you could omit the row and column labels as well as the dollar signs and write only the values as shown in Matrix 2. In fact, that’s the form that you would have to use if you were entering the matrix into most calculators with matrix capabilities. However, then you will have to remember that the rows represent the prices for the pizzas, drinks, and salads, while the columns represent the various pizza houses.

Matrix 2. The price matrix with labels removed.

After looking at your data, you might decide to drop Gina’s options since they are more expensive than any of the other pizza houses. If you do this, you will be left with a $3 \times 3$ square matrix.

Matrix 3. Square matrix after dropping the column with Gina’s prices.

If a matrix has only one column, it is called a column matrix. Matrix 4, which lists only the prices for Sal’s is a $3 \times 1$ column matrix. On the other hand, if you choose to look at the pizza prices alone, they can be represented by Matrix 5, which is a $1 \times 3$ row matrix.

Matrix 4. A $3 \times 1$ column matrix.

Matrix 5. A $1 \times 3$ row matrix.

1. Although matrices contain many data values, they can also be thought of as single entities. This feature allows us to refer to a matrix with a single capital letter.

A = 

\[
\begin{bmatrix}
12.16 & 10.10 & 10.86 & 10.65 \\
1.15 & 1.09 & 0.89 & 1.05 \\
4.05 & 3.69 & 3.69 & 3.85
\end{bmatrix}
\]

a. What is the value of $A_{31}$ and what does it represent?

b. What is the value of $A_{21}$? Of $A_{12}$? Of $A_{32}$?

c. Write an interpretation for entry $A_{21}$. For entry $A_{12}$. For entry $A_{32}$.

Pizza photograph: This image from pdphoto.org has been released into the public domain by its author and copyright holder, Jon Sullivan.

Salad photograph: This work is in the public domain in the United States because it is a work prepared by an officer or employee of the United States Government as part of that person’s official duties.

Definition: A square matrix has the same number of rows as columns.
2. As you continue to plan your pizza party, you discover that the local supermarket has a sale on 2-liter bottles of soft drinks. You decide not to order drinks from a pizza house after all. Write and label a $2 \times 3$ matrix that represents the prices for just pizza and salad at Vin's, Toni's and Sal's. Name this matrix $B$.

3. Suppose that when you were calling the pizza houses about prices, you also collected information about the cost of additional toppings and salad dressings. The costs for additional toppings were $1.15, $1.00, and $0.90 at Vin's, Toni's and Sal's, respectively. The costs for additional dressings were $0.45, $0.50, and $0.30 at Vin's, Toni's, and Sal's respectively. Represent this information in a $2 \times 3$ matrix whose rows represent the additional toppings and dressings and whose columns represent the three pizza houses. Label the rows and columns of your matrix. Name this matrix $C$.

4. Suppose that you want to find the cost of ordering pizzas with two toppings and salads with a choice of two salad dressings (the house dressing and one additional choice). This can be done by adding corresponding elements of your two price matrices from Exercises 2 and 3. By adding matrices $B$ and $C$, you can get a third matrix $D$.

$$D = B + C$$

Matrix $D$ will represent the total prices for two-topping pizzas and two-dressing salads at each pizza house.

a. What is the value of $D_{12}$? Of $C_{12}$? Of $D_{12}$?

b. Write an interpretation of $B_{12}$, $C_{12}$, and $D_{12}$.

c. Determine matrix $D$.

b. Apply the coupons to the prices from Exercise 4. In other words, determine matrix $F = D - E$.

6. Find the value of each of the following expressions. If it is not possible, explain.

a. $$\begin{pmatrix} 2 & 4 \\ -3 & 5 \end{pmatrix} + \begin{pmatrix} 7 & -8 \\ -1 & 2 \end{pmatrix}$$

b. $$\begin{pmatrix} 1 & 5 & 7 \\ 0 & 2 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 5 & 2 \end{pmatrix}$$

c. $$\begin{pmatrix} 1 & 0 \\ 3 & 0 \\ 5 & 0 \\ 0 & -4 \end{pmatrix} - \begin{pmatrix} 1 & 5 \\ -3 & 0 \\ 2 & -4 \\ 0 & -2 \end{pmatrix}$$

d. $$\begin{pmatrix} 1 & 5 & 10 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix}$$

7. The Matrix $M$ below shows the mileage between 5 major U.S. cities.

$$M = \begin{pmatrix}
    Atlanta & Boston & Chicago & Los Angeles & St. Louis \\
    0 & 1,075 & 716 & 2,211 & 555 \\
    1,075 & 0 & 1,015 & 3,026 & 1,187 \\
    716 & 1,015 & 0 & 2,034 & 297 \\
    2,211 & 3,026 & 2,034 & 0 & 1,842 \\
    555 & 1,187 & 297 & 1,842 & 0
\end{pmatrix}$$

a. Entries that are located in row $i$, column $j$, where $i = j$, are said to be located on the main diagonal of the matrix. Examine the entries of the main diagonal of $M$. What do you notice?

b. A square matrix $R$ with dimension $n \times n$ is symmetric if $R_{ij} = R_{ji}$, where $i$ and $j = 1, 2, 3, \ldots n$. Is matrix $M$ symmetric? Explain.

c. Give an example of a $3 \times 3$ matrix that is symmetric and has entries on the main diagonal that differ from 0.

d. Could a matrix that is not square be symmetric? Why?

**TI-84 Instructions**

A TI-84 graphing calculator has matrix capabilities. Here’s how to enter a matrix into a TI-84:

- Press 2nd MATRIX (x²-key) and highlight Edit.
- Press the number corresponding to the letter that you want to name your matrix.
- Enter the number of rows, press ENTER, then the number of columns, and press ENTER.
• Enter the matrix entries row by row, pressing ENTER after each entry.
• When you have completed entering your matrix, press 2nd QUIT (MODE-key).
• To view your matrix, press 2nd MATRIX, the number corresponding to the name of your matrix, and then ENTER.

8. Enter the following matrices into your calculator.
\[
A = \begin{bmatrix} 2 & 4 \\ 0 & -4 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 \\ 2 & 7 \\ 5 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}
\]

9. a. Use your calculator to determine \( A + B \). Here’s how:
• Press 2nd MATRIX, enter the number corresponding to where you stored matrix \( A \).
• Press +.
• Press 2nd MATRIX, enter the number corresponding to where you stored matrix \( B \).
• Press ENTER and read the result from your screen.
b. Use your calculator to determine \( A - B \).
c. What happens if you try to use your calculator to find \( A + C \)?

10. A name brand store has three outlets. Each outlet carries jackets, shirts, and pants. Matrix \( N \) shows the sales for November and matrix \( D \), the sales for December.

\[
N = \begin{bmatrix} 145 & 173 & 125 \\ 75 & 148 & 96 \\ 95 & 210 & 222 \end{bmatrix}, \quad D = \begin{bmatrix} 210 & 188 & 157 \\ 121 & 172 & 143 \\ 126 & 259 & 277 \end{bmatrix}
\]

a. Enter matrices \( N \) and \( D \) into your calculator.
b. Using your calculator, determine the total sales of jackets, shirts, and pants at each outlet for the combined months of November and December.
c. Matrix \( R \) shows the January returns of items purchased in December. Determine the actual sales for December after accounting for the returns. In other words, determine \( D - R \).

\[
R = \begin{bmatrix} 101 & 25 & 37 \\ 43 & 22 & 30 \\ 24 & 117 & 124 \end{bmatrix}
\]

11. The National League batting leaders for 2011 had the following batting statistics.

<table>
<thead>
<tr>
<th>Player</th>
<th>AB</th>
<th>R</th>
<th>H</th>
<th>HR</th>
<th>RBI</th>
<th>Avg</th>
</tr>
</thead>
<tbody>
<tr>
<td>J. Reyes (New York)</td>
<td>537</td>
<td>101</td>
<td>181</td>
<td>7</td>
<td>44</td>
<td>0.337</td>
</tr>
<tr>
<td>R. Braun (Milwaukee)</td>
<td>563</td>
<td>109</td>
<td>187</td>
<td>33</td>
<td>111</td>
<td>0.332</td>
</tr>
<tr>
<td>M. Kemp (Los Angeles)</td>
<td>602</td>
<td>115</td>
<td>195</td>
<td>39</td>
<td>126</td>
<td>0.324</td>
</tr>
</tbody>
</table>

The following statistics for the same three players were published at the end of the 2012 season.

<table>
<thead>
<tr>
<th>Player</th>
<th>AB</th>
<th>R</th>
<th>H</th>
<th>HR</th>
<th>RBI</th>
<th>Avg</th>
</tr>
</thead>
<tbody>
<tr>
<td>J. Reyes (New York)</td>
<td>642</td>
<td>86</td>
<td>184</td>
<td>11</td>
<td>57</td>
<td>0.287</td>
</tr>
<tr>
<td>R. Braun (Milwaukee)</td>
<td>598</td>
<td>108</td>
<td>191</td>
<td>41</td>
<td>112</td>
<td>0.319</td>
</tr>
<tr>
<td>M. Kemp (Los Angeles)</td>
<td>403</td>
<td>74</td>
<td>122</td>
<td>23</td>
<td>69</td>
<td>0.303</td>
</tr>
</tbody>
</table>

Find and label a matrix that displays the changes in these statistics from the 2011 season to the 2012 season. Notice that several of the statistics decreased from 2011 to 2012. How will you show this in your matrix?

Activity 2: Matrix Multiplication Served With Salad and Pizza

Return to the pizza problem from Activity 1. In that activity you used the price matrix shown in Matrix 6 to represent the total costs of two-topping pizzas and salads with a choice of two dressings from Vin’s, Toni’s, and Sal’s.

\[
D = \begin{bmatrix} \text{Pizza} \\ \text{Salad} \end{bmatrix} = \begin{bmatrix} 11.25 & 11.96 & 11.90 \\ 3.69 & 4.14 & 4.35 \end{bmatrix}
\]

**Matrix 6. Costs of pizzas and salads.**

Suppose you wish to order four pizzas and four salads and want to compare the prices at the three pizza houses. To do this, multiply each element in matrix \( D \) by 4 to get a new matrix that is equal to \( 4D \).

\[
4D = 4 \times \begin{bmatrix} 11.25 & 11.96 & 11.90 \\ 3.69 & 4.14 & 4.35 \end{bmatrix} = \begin{bmatrix} 4(11.25) & 4(11.96) & 4(11.90) \\ 4(3.69) & 4(4.14) & 4(4.35) \end{bmatrix} = \begin{bmatrix} 45.00 & 47.84 & 47.60 \\ 14.76 & 16.56 & 17.40 \end{bmatrix}
\]

If you call the new matrix \( T \) and label the rows and columns of the matrix, you have

\[
T = \begin{bmatrix} \text{Pizza} & \text{Salad} \\ \text{Vin’s} & $45.00 & $47.84 & $47.60 \\ \text{Toni’s} & $14.76 & $16.56 & $17.40 \end{bmatrix}
\]

**Matrix 7. Costs of 4 pizzas and 4 salads.**

When working with matrices, a real number is often called a scalar. In the multiplication above, the scalar is 4.

**Scalar Multiplication**

If \( k \) is a real number and \( A \) is a matrix, the matrix \( kA \) is formed by multiplying each entry in matrix \( A \) by \( k \).

1. Suppose instead you wish to order six pizzas and six salads and want to compare the costs at the three pizza houses. Find the matrix \( 6D \).
stock up his locker for between-class snacks. He chooses four small bags of chips, five candy bars, a box of cheese crackers, three packs of sour drops, and two bags of cookies. Ruben’s purchases can be represented by the row matrix \( Q \) (Matrix 8).

\[
Q = \begin{bmatrix}
4 & 5 & 1 & 3 & 2 \\
\end{bmatrix}
\]

**Matrix 8. Quantity matrix for Ruben’s purchases.**

Suppose further that chips cost 30 cents a bag, candy bars cost 35 cents each, crackers cost 50 cents a box, sour drops cost 20 cents a pack and cookies sell for 75 cents a bag. These prices are represented by the column matrix \( P \) (Matrix 9).

\[
P = \begin{bmatrix}
30 \\
35 \\
50 \\
20 \\
75 \\
\end{bmatrix}
\]

**Matrix 9. Price matrix for Ruben’s purchases.**

Now the obvious question to ask is, “How much did Ruben pay for all these snacks?” You can answer this question by multiplying the price matrix \( P \) by the quantity matrix \( Q \).

\[
QM = \begin{bmatrix}
4 & 5 & 1 & 3 & 2 \\
\end{bmatrix} \begin{bmatrix}
30 \\
35 \\
50 \\
20 \\
75 \\
\end{bmatrix} = 4(30) + 5(35) + 1(50) + 3(20) + 2(75) \\
= 120 + 175 + 50 + 60 + 150 \\
= 555 \text{ cents} = $5.55
\]

This matrix computation shows exactly what the clerk at the store would do to figure out Rubin’s bill. The price of each item is multiplied by the number purchased and then products are summed. In other words, the first entry in the row matrix multiplies the first entry in the column matrix, the second entry in the row matrix multiplies the second entry in the column matrix, . . . , and the last entry in the row matrix multiplies the last entry in the column matrix. Then these products are summed to yield a final result, which turns out to be a single number or a \( 1 \times 1 \) matrix.

2. A second student, Terri, goes along with Ruben to the store. Terri’s purchases are a bag of chips, two candy bars, two packs of gum that cost 25 cents each, and a medium drink for 75 cents.

a. Set up a row matrix \( Q \) for the quantities and a column matrix \( P \) for the costs. Label these matrices similarly to how Matrix 8 and Matrix 9 were labeled.

\[
Q = \begin{bmatrix}
1 \\
2 \\
2 \\
\end{bmatrix}
\]

\[
P = \begin{bmatrix}
25 \\
35 \\
25 \\
75 \\
\end{bmatrix}
\]

**Matrix 10. Quantity matrix for pizza house purchases.**

To find the answer to this problem, we will multiply matrix \( D \) (Matrix 6) by \( Q \). In order to calculate \( QD \), we will apply the process used to multiply a column matrix by a row matrix to each column of matrix \( D \).
c. Could you apply the process described in (a) to multiply a 1 × 3 row matrix A and a 3 × 2 matrix B? If so, what would be the dimension of the product AB? If not, explain why not.

b. Could you apply the process described in (a) to multiply a 1 × 3 row matrix A and a 3 × 2 matrix B? If so, what would be the dimension of the product AB? If not, explain why not.

c. Could you apply the process described in (a) to multiply a 1 × 4 row matrix A and a 3 × 2 matrix B? If so, what would be the dimension of the product AB? If not, explain why not.

In general, when you multiply a 1 × k row matrix A times a k × n matrix B, the product AB has dimension 1 × n.

Diagram 1 shows how the dimensions of matrices A and B fit together to form a matrix product AB.

\[ A \times B = AB \]

5. Find the product of V and W.

\[ V = \begin{bmatrix} 2 & 4 & 7 & 0 & 1 \end{bmatrix}, \quad W = \begin{bmatrix} 1 & 2 \\ 3 & 8 \\ 2 & 5 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} \]

Up to this point, we have multiplied a 1 × 2 row matrix and a 2 × 3 price matrix to determine the total cost of 5 pizzas and 3 salads at each pizza house. But what if you want to compare the total costs of several different combinations of pizzas and salads? The options are spelled out in a new quantity matrix B (Matrix 12).

If you multiply matrix B times matrix D (Matrix 6), the product will be a 3 × 3 matrix. Here is how to adapt the process used to find the product Q × D shown in Matrix 11 to determine the product B × D:

\[ B \times D = \begin{bmatrix} 4 & 3 \\ 4 & 4 \\ 5 & 3 \end{bmatrix} \times \begin{bmatrix} 11.25 & 11.96 & 11.90 \\ 3.69 & 4.14 & 4.35 \end{bmatrix} = \begin{bmatrix} 4(11.25) + 3(3.69) & 4(11.96) + 3(4.14) & 4(11.90) + 3(4.35) \\ 4(11.25) + 4(3.69) & 4(11.96) + 4(4.14) & 4(11.90) + 4(4.35) \\ 5(11.25) + 3(3.69) & 5(11.96) + 3(4.14) & 5(11.90) + 3(4.35) \end{bmatrix} \]

\[ = \begin{bmatrix} 45.00 + 11.07 & 47.84 + 12.42 & 47.60 + 13.05 \\ 45.00 + 14.76 & 47.84 + 16.56 & 47.60 + 17.40 \\ 56.25 + 11.07 & 59.80 + 12.42 & 59.50 + 13.05 \end{bmatrix} \]

\[ = \begin{bmatrix} 56.07 & 60.26 & 60.65 \\ 59.76 & 64.40 & 65.00 \\ 67.32 & 72.22 & 72.55 \end{bmatrix} \]

The labels of the product BD are shown in Matrix 13.

\[ E = BD = \begin{bmatrix} 56.07 & 60.26 & 60.65 \\ 59.76 & 64.40 & 65.00 \\ 67.32 & 72.22 & 72.55 \end{bmatrix} \]

It is important to observe that the dimension of the product can also be described by using the row and column labels. Matrix B classifies the data according to Options (rows) and Foods (columns). Hence, you can refer to matrix B as an Options by Foods matrix. Likewise you can describe D as a Foods by Houses matrix. The product B × D, in turn, results in a matrix of dimension Options by Houses as is shown by the following diagram.

\[ B \times D = E \]

6. In Matrix E (Matrix 13), entry \( E_{23} \) represents the cost of Option 1, the purchase of four pizzas and three salads, at Vin’s. How would you interpret \( E_{23} \) and \( E_{32} \)?
In order for the product of two matrices to be defined, the number of columns in the first matrix must equal the number of rows in the second matrix. If A has dimension $m \times k$ and B has dimension $k \times n$, then the product $P = AB$ has dimension $m \times n$ as shown by Diagram 3.

$$A \times B = AB$$

![Diagram 3. The dimensions of the product of two matrices.](image)

### Activity 2

7. Suppose that $M$ is a $2 \times 3$ matrix and $N$ is a $4 \times 2$ matrix. Which of the products is defined, $MN$ or $NM$? Explain.

8. Consider the two matrices below.

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 & 1 \\ 1 & 1 & 4 & 2 & 0 \\ 3 & 2 & 2 & 1 & 6 \\ 2 & 8 & 3 & 1 & 6 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

a. What is the dimension of the product $P = A \times B$?

b. What is the entry in row 2 and column 3 of the product matrix $P$? In other words, determine $P_{23}$.

c. What is the entry in row 4 and column 1 of the product $P$? In other words, determine $P_{41}$.

d. Multiply matrix $Q$ times matrix $C$ to get a matrix $R$. Label the rows and columns of matrix $R$.

e. Interpret $R_{12}$, $R_{21}$, and $R_{23}$.

9. Use the following matrices to compute the given expression. If the expression is not defined, give the reason.

$$A = \begin{bmatrix} 3 & 8 & -1 \\ 2 & 0 & 4 \end{bmatrix}, B = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & .5 \\ 1 & 5 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix}$$

a. $4A$

b. $AB$

c. $BA$

d. $CA$

e. $C + A$

10. Massachusetts imposes a sales tax of 6.25% on meals whether they are eaten in a restaurant or taken out. How could you use matrices (scalar multiplication, matrix multiplication and/or matrix addition) to determine a matrix of the total costs (price plus tax) of the three options at the three pizza houses (see Matrix 13) if they were located in Massachusetts? Use your method to determine a total costs matrix $T$.

11. Rosa and Max go out to eat at Sammy’s Drive Inn. Rosa orders a Sammy’s special, fries, and a shake. Max has a cheeseburger, a baked potato with sour cream, and a shake. The approximate number of calories, grams of fat, and milligrams of cholesterol in each of these foods are represented in the following table.

<table>
<thead>
<tr>
<th>Item</th>
<th>Calories</th>
<th>Fat (g)</th>
<th>Cholesterol (mg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cheeseburger</td>
<td>450</td>
<td>40</td>
<td>50</td>
</tr>
<tr>
<td>Sammy’s special</td>
<td>570</td>
<td>48</td>
<td>90</td>
</tr>
<tr>
<td>Potato/sour cream</td>
<td>500</td>
<td>45</td>
<td>25</td>
</tr>
<tr>
<td>French fries</td>
<td>300</td>
<td>30</td>
<td>0</td>
</tr>
<tr>
<td>Shake</td>
<td>400</td>
<td>22</td>
<td>50</td>
</tr>
</tbody>
</table>

- a. Write a matrix $Q$ that describes Rosa’s and Max’s orders, with the columns representing the foods. Label the rows and columns of this matrix.
- b. Write a matrix $C$ that represents the information in the preceding table with the rows representing the foods. Label the rows and columns of this matrix.
- c. What are the dimensions of matrices $Q$ and $C$?
- d. What is the dimension of the product $Q \times C$? Show why your answer is correct by using a diagram similar to the one shown in Diagram 3.
- e. The dimension of matrix $Q$ could be described as Persons by Foods. Describe the dimensions of matrices $C$ and $Q \times C$. Justify your answer for matrix $Q \times C$ with a diagram similar to Diagram 2.
- f. Multiply matrix $Q$ times matrix $C$ to get a matrix $R$. Label the rows and columns of matrix $R$.
- g. Interpret $R_{12}$, $R_{21}$, and $R_{23}$.

**TI-84 Instructions and Properties of Matrices**

In Activity 1 you learned how to enter matrices into your calculator. Enter the following matrices into your calculator.

$$A = \begin{bmatrix} -1 & 5 & 2 \\ -3 & 0 & 4 \\ -2 & 1 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 4 & 6 & 3 \\ 0 & 2 & 4 \\ 5 & 7 & 3 \end{bmatrix}$$

**Matrix 14. Three square matrices.**

12. From Activity 1, you know how to add two matrices. Now, you will use your calculator to multiply two matrices. Determine $AB$. Here’s how:

- Press 2nd MATRIX, enter the number corresponding to where you stored matrix $A$. 

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13. **HiMAP Pull-Out Section: Fall/Winter 2014**
Activity 3: Population Growth: The Leslie Model

You will want to use a calculator or computer software with matrix capabilities for this activity. After completing Activities 1 and 2, you should be able to add, subtract, and multiply matrices. In this activity, you will use these skills to model the growth of a population.

Population growth is a topic that is of great concern to many people. For example, urban planners are interested in knowing how many people there will be in various age groups after certain periods of time have passed. Wildlife managers are concerned about keeping animal populations at levels that can be supported in their natural habitats.

If you know the age distribution of a population at a certain date and the birth and survival rates for age-specific groups, you can use these data to create a mathematical model. Then, you can use your model to determine the age distributions of the survivors and descendants of the original population at successive intervals of time.

The problem used to illustrate this model was posed by P. H. Leslie. In his problem, the growth rate of a population of an imaginary species of small brown rats, Rattus norvegicus, is examined. In order to simplify the model the following assumptions are made.

• Only the female population is considered.
• Birth rates and survival rates are held constant over time.
• The survival rate of a rat is the probability that it will survive and move into the next age group.
• The lifespan of these rodents is 15 – 18 months.
• The rats will have their first litter at approximately 3 months and continue to reproduce every 3 months until they reach the age of 15 months.

Birth rates and age-specific survival rates for 3-month periods are summarized in Table 1.

<table>
<thead>
<tr>
<th>Age (months)</th>
<th>Birth Rate</th>
<th>Survival Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 - 3</td>
<td>0</td>
<td>0.6</td>
</tr>
<tr>
<td>3 - 6</td>
<td>0.3</td>
<td>0.9</td>
</tr>
<tr>
<td>6 - 9</td>
<td>0.8</td>
<td>0.9</td>
</tr>
<tr>
<td>9 - 12</td>
<td>0.7</td>
<td>0.8</td>
</tr>
<tr>
<td>12 - 15</td>
<td>0.4</td>
<td>0.6</td>
</tr>
<tr>
<td>15 - 18</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1. Birth/Survival rates for each age group.
Suppose the original female rat population is 42 animals with the age distribution given in Table 2.

<table>
<thead>
<tr>
<th>Age (months)</th>
<th>0 - 3</th>
<th>3 - 6</th>
<th>6 - 9</th>
<th>9 - 12</th>
<th>12 - 15</th>
<th>15 - 18</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number</td>
<td>15</td>
<td>9</td>
<td>13</td>
<td>5</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Table 2. Number in Female Rat Population Broken Down by Age.**

You can use Table 2 and the birth rate/survival rate information in Table 1 to find the total number of rats and their age distribution after 3 months (1 cycle). To find this new distribution, you will need to find two things:

A. the number of new female babies introduced into the population.

B. the number of female rats that survive in each group and move up to the next age group.

We begin by tackling Item A. To find the number of new births after 3 months (1 cycle), we need to multiply the number of female rates in each age group times the corresponding birth rates and then find the sum.

1. Going back to Activity 2, you learned how to accomplish the type of calculations described above using matrix multiplication. To find the number of new female babies introduced into the population (Item A), multiply a row matrix of the age distribution times a column matrix of the birth rates. In other words, determine the following product. (Do not round your answer.)

\[
\begin{bmatrix}
15 \\
9 \\
13 \\
5 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
0.3 \\
0.8 \\
0.7 \\
0.4 \\
0
\end{bmatrix}
\]

So, based on the answer to Exercise 1, after 3 months there will be about 17 female rats in the 0 – 3 age group.

Next, we work on Item B. Table 3 shows the calculations for the number of female rats who survive in each age group and move up to the next.

<table>
<thead>
<tr>
<th>Age</th>
<th>Number</th>
<th>Survival Rate</th>
<th>Number Moving Up to the Next Age Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 - 3</td>
<td>15</td>
<td>0.6</td>
<td>(15)(0.6) = 9.0 move up to the 3 - 6 age group.</td>
</tr>
<tr>
<td>3 - 6</td>
<td>9</td>
<td>0.9</td>
<td>(9)(0.9) = 8.1 move up to the 6 - 9 age group.</td>
</tr>
<tr>
<td>6 - 9</td>
<td>13</td>
<td>0.9</td>
<td></td>
</tr>
<tr>
<td>9 - 12</td>
<td>5</td>
<td>0.8</td>
<td></td>
</tr>
<tr>
<td>12 - 15</td>
<td>0</td>
<td>0.6</td>
<td>No resident lives beyond 18 months.</td>
</tr>
<tr>
<td>15 - 18</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

**Table 3. Calculations for Item B.**

2. Make a copy of Table 3 and complete the calculations for the last column.

The distribution of female rats after 3 months (1 cycle) is shown in Table 4. You should verify these entries match with your answers to Exercises 1 and 2.

<table>
<thead>
<tr>
<th>Age</th>
<th>0 - 3</th>
<th>3 - 6</th>
<th>6 - 9</th>
<th>9 - 12</th>
<th>12 - 15</th>
<th>15 - 18</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number</td>
<td>16.6</td>
<td>9</td>
<td>8.1</td>
<td>11.7</td>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>

**Table 4. The Number of Female Rats in Each Age Category after 3 Months (1 Cycle).**

The sum of the number of female rats in each age group results in a total population of female rats equal to 16.6 + 9.0 + 8.1 + 11.7 + 4.0 + 0, or 49.4. So, after 3 months (1 cycle) the female rat population has grown from 42 to approximately 50. Notice that the numbers in Table 4 were not rounded to the nearest integer. This is because when the values are to be used for further analysis, rounding off can mean a significant difference in calculations over time even though it makes no sense to have a fractional part of a rat.

3. Use the distribution in Table 4, the female rat population after 3 months, and the process introduced in Exercises 1 and 2 to compute the following.

a. Calculate the number of newborn rats (aged 0 – 3) after 6 months (2 cycles). Show how you can use matrix multiplication to solve this problem.

b. Calculate the number of rats that survive in each age group after 6 months and move up to the next age group by completing a table similar to Table 3.

c. Use the results in (a) and (b) to show the distribution of the female rat population after 6 months. Approximately how many rats will there be after 6 months (2 cycles)?

d. Use your population distribution from part (c) to calculate the number of female rats in each age group and the approximate total number after 9 months (3 cycles).

4. Explore the possibility of multiplying the initial population in a row matrix times some column matrix to find the number of rats after 3 months (1 cycle) that move from:

a. The 0 – 3 age group to the 3 – 6 age group. (Hint: The column matrix that you use will need to contain some zeros in order to produce the desired product.)

b. The 3 – 6 age group to the 6 – 9 age group.

c. The 6 – 9 age group to the 9 – 12 age group.

d. The 9 – 12 age group to the 12 – 15 age group.

e. The 12 – 15 age group to the 15 – 18 age group.

Thus far, you have found that it is possible to use an initial population distribution along with birth and survival...
rates to predict the population numbers at future times. As you explored your model, you found that you could look 2 or 3 cycles into the future. However, the arithmetic soon became cumbersome. What do wildlife managers and urban planners do if they want to look 10, 20, or even more cycles into the future?

In Exercise 4, you began to get a glimpse of the model that Leslie proposed. The use of matrices seems to hold the key. And with the aid of computer software or a calculator, looking ahead many cycles is not difficult.

Return to the original rat model. If you multiply the original population distribution \( P_0 \) times a matrix that we call \( L \), you can calculate the population distribution at the end of cycle 1 \( (P_1) \).

\[
P_L = \begin{bmatrix} 15 & 9 & 13 & 5 & 0 \end{bmatrix}
\]

\[
L = \begin{bmatrix} 0 & 0.6 & 0 & 0 & 0 \\ 0.3 & 0 & 0.9 & 0 & 0 \\ 0.8 & 0 & 0.9 & 0 & 0 \\ 0.7 & 0 & 0 & 0.8 & 0 \\ 0.4 & 0 & 0 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

The matrix \( L \) is called the Leslie matrix. This matrix is formed by joining the column matrix containing the birth rates of each age group and a series of column matrices that contain the survival rates. Notice that the survival-rate columns contain the survival rates as one entry and zeros everywhere else. The survival rates (of which there is one less than the actual number of survival rates since no animal survives beyond the 15 – 18 age group) lie along the super diagonal that is immediately above the main diagonal of the matrix. (The entries on the main diagonal are the entries in row \( i \) column \( i \), or \( L_{ii} \).)

5. a. Use the Leslie matrix to calculate the population distribution at the end of cycle 2 \( (P_2) \) by determining \( P_1L \). Check that the results match with the population distribution that you determined in Exercise 3(c).

b. Use the Leslie matrix to calculate the population distribution at the end of cycle 3 \( (P_3) \) by determining \( P_2L \). Check that the results match with the population distribution that you determine in Exercise 3(d).

When the matrix \( L \) is multiplied by a population distribution \( P_k \), a new population distribution \( P_{k+1} \) results. To find the population distribution at the end of other cycles, the process can be continued.

\[
P_1 = P_0L
\]

\[
P_2 = P_1L = (P_0L)L = P_0(LL) = P_0L^2
\]

In general, \( P_k = P_0L^k \).

6. Use the formula \( P_k = P_0L^k \) to find the population distribution for the rats after 24 months (8 cycles) and the total population of the rats.

7. a. Up to this point, you have determined the total population by summing the entries in the population distribution for age groups. Return to matrix \( P_1 = \begin{bmatrix} 16.6 & 9.0 & 8.1 & 11.7 & 4.0 & 0 \end{bmatrix} \). Find a column matrix \( S \) with the property that \( P_1S \) is the sum of the entries in matrix \( P_1 \).

b. Find the total female population after 30 months (10 cycles) by determining \( P_{10}S \) = \( (P_0L^{10})S \).

8. Suppose the Rattus norvegicus start dying off from overcrowding when the total female population for a colony reaches 250. Find how long it will take for this to happen when the initial population is

a. \([ 18 \ 9 \ 7 \ 0 \ 0 \ 0 ]\)? (Hint: If using a TI-84, start with 10 cycles. Then press 2nd ENTRY (ENTRY-key) and change the power. Use guess and check to determine an answer.)

b. \([ 25 \ 15 \ 10 \ 11 \ 7 \ 13 ]\)?

9. Consider a species of deer with birth and survival rates given in Table 5. Notice that in this case 1 cycle = 2 years.

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>Birth Rate</th>
<th>Survival Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 - 2</td>
<td>0</td>
<td>0.6</td>
</tr>
<tr>
<td>2 - 4</td>
<td>0.8</td>
<td>0.8</td>
</tr>
<tr>
<td>4 - 6</td>
<td>1.7</td>
<td>0.9</td>
</tr>
<tr>
<td>6 - 8</td>
<td>1.7</td>
<td>0.9</td>
</tr>
<tr>
<td>8 - 10</td>
<td>0.8</td>
<td>0.7</td>
</tr>
<tr>
<td>10 - 12</td>
<td>0.4</td>
<td>0.3</td>
</tr>
</tbody>
</table>

a. Construct the Leslie matrix for this animal.

b. Given that \( P_0 = \begin{bmatrix} 50 & 30 & 24 & 24 & 12 & 8 \end{bmatrix} \), use \( P_0S \) to calculate the initial total number of female deer. (Use the same matrix \( S \) that you determined in 7(a).

c. Determine the total number of female deer after 10 years (5 cycles).

d. Suppose the natural range for this animal can sustain a herd that contains a maximum of 1,250 females. How long before this herd size is reached?

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The Teachers College Mathematical Modeling Handbook II: The Assessments, provides examples of new approaches to assessment designed especially for the twenty six mathematical modeling activities of Handbook I. Handbook II includes readiness tests for each module, references to the CCSM standards for which the module is appropriate, a final test for the module that can be utilized in whole or in part by classroom teachers, model-eliciting activities utilizing insights gained from the corresponding module of Handbook I, and, finally, extra-class activities and answer keys for the instruments in the assessment modules itself. To assist teachers in implementing assessment, all test instruments are provided in a black-line master format for duplication and distribution by the teacher.

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