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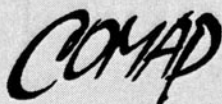
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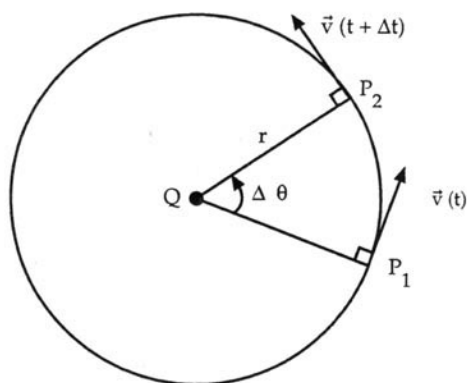
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Module 507

The Shape of the Surface of a Rotating Liquid

Anthony L. Peressini



Applications of Elementary Calculus
to Physics

MODULES AND MONOGRAPHS IN UNDERGRADUATE
MATHEMATICS AND ITS APPLICATIONS (UMAP) PROJECT

The goal of UMAP was to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications to be used to supplement existing courses and from which complete courses may eventually be built.

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THE SHAPE OF THE SURFACE
OF A ROTATING LIQUID*

by

Anthony L. Peressini
Department of Mathematics
University of Illinois
Urbana, Illinois 61801

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Title: THE SHAPE OF THE SURFACE OF A ROTATING FLUID

Author: Anthony L. Peressini
Department of Mathematics
University of Illinois
Urbana, IL 61801

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Classification: APPL ELEM CALC/PHYSICS

Prerequisite Skills:

1. The interpretation of the derivative as velocity and acceleration and the algebra of vectors in the plane. The exercises require the computation of a volume of revolution and the use of the limit

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1.$$

Output Skills:

1. Be able to analyze the forces on a liquid rotating in a vertical cylinder at constant angular speed to show that the surface of the liquid is a paraboloid of revolution.

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1. THE BASIC PROBLEM

Suppose that a cylindrical can partly filled with a liquid is placed so that its central axis is vertical. If we begin to spin the can around its axis at a constant speed, then the can will impart a rotational motion to the liquid around the same axis. After awhile, the liquid in the can will be rotating so that it is essentially motionless relative to the can. The surface of the liquid will be convex as indicated in Figure 1 because the centrifugal force on the liquid particles increases with the distance from the axis of the can.

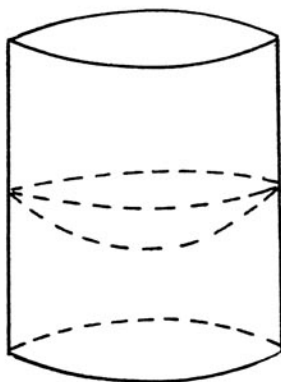


Figure 1. The surface of a liquid in a rotating cylinder is convex.

It is apparent that the surface of the liquid is symmetric to the axis of rotation so that it may be regarded as the surface of revolution of a plane curve. But what plane curve? We will show that this curve is a parabola so that the liquid surface is a paraboloid of revolution around the axis of rotation. You will see that this derivation is a simple but rather interesting application of Newton's Second Law of Motion and the interpretation of velocity and acceleration as the first and second derivatives of the position function.

2. THE MATHEMATICAL MODEL

2.1 The Simplifying Assumptions

In our discussion of the shape of the surface of the rotating liquid, we will make the following simplifying assumptions:

- a) The effects of surface tension of the liquid will not be considered. Surface tension causes the liquid to either "crawl up" the walls of the container like water or to "crawl down" the walls like mercury. (This effect is easily observed in the surface of water in a glass.)
- b) All of the particles of the liquid and the container are rotating with the same angular speed ω around the central axis of the container.
- c) The only forces considered on the particles of the liquid are the gravitational force and the centrifugal force.

2.2 Computing the Forces

To compute the forces under consideration, we consider a plane containing the axis of rotation and introduce a rectangular coordinate system with the axis of rotation as the y-axis and the origin at the surface of the rotating liquid. (See Figure 2). A mass particle P on the surface of the liquid is acted on by the force \vec{F}_G of gravity and the centrifugal force \vec{F}_C resulting from the particle's rotation. Since we are assuming that the particle P is motionless relative to the other particles in the liquid, it follows that the resultant force $\vec{F} = \vec{F}_G + \vec{F}_C$ must have a direction perpendicular to the surface of the liquid at P. Otherwise, this resultant force would have a non-zero component in a direction tangent to the surface and this component would cause P to move relative to the other liquid particles and the container.

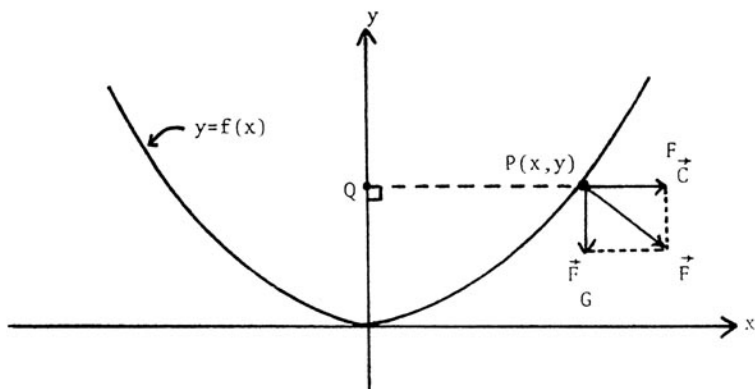


Figure 2. The surface of the rotating liquid is described in the coordinate system as the curve $y=f(x)$.

According to Newton's Second Law of Motion, the gravitational force \vec{F}_G on P is

$$(1) \quad \vec{F}_G = m\vec{g},$$

where m is the mass of P and \vec{g} is the acceleration of gravity. On the surface of the Earth, the magnitude of \vec{g} is 980 cm/sec^2 or 32 ft./sec^2 . The direction of \vec{F}_G is toward the center of the Earth, that is straight downward in Figure 2.

2.3 The Centripetal Force

Now let us turn to the discussion of the force \vec{F}_C . If we imagined the mass particle P to be attached to the axis of rotation by a string, then it would be necessary for the string to exert a force of constant magnitude on P, directed perpendicularly toward the axis, in order to maintain the rotation of P around the axis at a constant angular speed. This is a consequence of Newton's First Law of Motion which states that P would travel in a straight line at a constant speed or remain at rest if no outside force acts on P. This force is called *centripetal force*. The force of equal magnitude and opposite direction exerted by the mass particle P on the string according to Newton's Third Law of Motion is called *centrifugal force*.

Newton's Second Law of Motion also implies that the centrifugal force \vec{F}_C on P is given by

$$(2) \quad \vec{F}_C = -m \vec{a}_C,$$

where m is the mass of P and \vec{a}_C is the centripetal acceleration. But how can we determine the centripetal acceleration \vec{a}_C ? First of all, we will show that the direction of \vec{a}_C is *central*, that is, from P perpendicularly toward the axis. For P is rotating in a circular path of radius r and center at the point Q of the axis of rotation. (See Figures 2 and 3.)

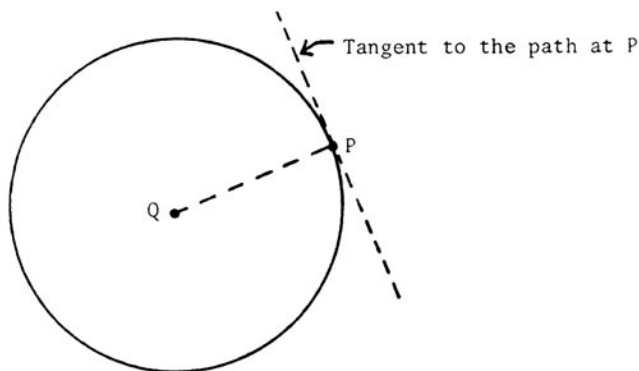


Figure 3. When the liquid rotates at a constant angular speed, the acceleration of a mass at P is centripetal: its direction is the direction from the axis of rotation to the point P.

If the centripetal acceleration \vec{a}_C were not parallel to \vec{PQ} , then it would have a non-zero component in a direction tangent to the circular path at P. This component would speed up or slow down P, contrary to the fact that the speed of P is constant. Therefore, \vec{a}_C and \vec{F}_C are parallel to \vec{QP} .

To compute the magnitude of \vec{a}_C we proceed as follows: Suppose that in the time interval between t and $t+\Delta t$, the position of P changes from $P_1 = P(t)$ to $P_2 = P(t+\Delta t)$, the angle swept out by \vec{OP} changes by $\Delta\theta$, and the velocity vector

\vec{v} changes from $\vec{v}(t)$ to $\vec{v}(t+\Delta t)$, as shown in Figure 4.

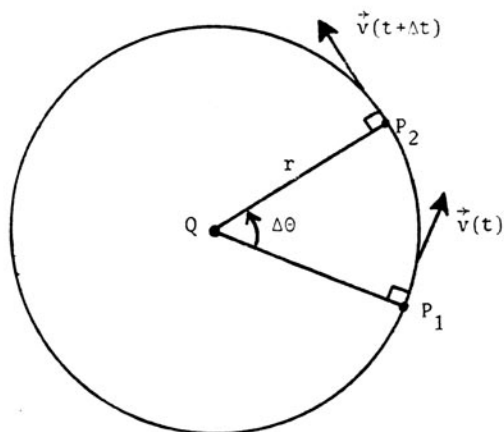


Figure 4. The change in the velocity vector \vec{v} in a time increment Δt .

Since the velocity vector \vec{v} is always tangent to the path at P, it remains perpendicular to the position vector \vec{QP} . Also, the magnitude $|\vec{v}|$ of \vec{v} , that is, the speed of P, remains constant as P rotates around Q. Therefore, if $\Delta\vec{v} = \vec{v}(t + \Delta t) - \vec{v}(t)$, then the vector triangle in Figure 5

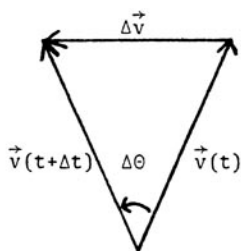


Figure 5. When the vectors $\vec{v}(t)$ and $\vec{v}(t+\Delta t)$ of Figure 4 are drawn to emanate from the same point, they make a triangle similar to triangle QP_1P_2 in Figure 4.

is similar to the triangle QP_1P_2 in Figure 4, and $|\vec{v}| = |\vec{v}(t)| = |\vec{v}(t + \Delta t)|$. Consequently, we obtain the equation

$$(3) \quad \frac{|\Delta\vec{v}|}{|\vec{v}|} = \frac{\text{length of chord } P_1P_2}{r}$$

by equating ratios of the lengths of corresponding sides of these similar triangles. For small values of Δt , the length of the chord P_1P_2 is nearly the same as the length of the arc of the circle from P_1 to P_2 , and this latter length is $|\vec{v}|\Delta t$. Consequently, Equation (3) can be replaced with the following approximation

$$(4) \quad |\Delta\vec{v}| \doteq \frac{|\vec{v}|^2 \Delta t}{r},$$

with the approximation approaching precision as $\Delta t \rightarrow 0$. (See Exercise 2.) We can now compute the magnitude of the centripetal acceleration as follows:

$$(5) \quad |\vec{a}_C| = \lim_{\Delta t \rightarrow 0} \frac{|\Delta\vec{v}|}{\Delta t} = \frac{|\vec{v}|^2}{r}.$$

Since $|\vec{v}| = r\omega$ where ω is the constant angular speed of rotation, we can express $|\vec{a}_C|$ as follows

$$(6) \quad |\vec{a}_C| = r\omega^2.$$

Thus, we have shown that *the centripetal force on a particle P of mass m located at a distance r from the axis of rotation is a central force with magnitude $mr\omega^2$ where ω is the constant angular speed of rotation.*

2.4 The Surface is Identified

We are now in a position to identify the function $y = f(x)$ in Figure 2 whose graph generates the surface of the liquid when it is rotated around the y -axis. Since P is motionless relative to the other particles of liquid,

the vector $\vec{F} = \vec{F}_G + \vec{F}_C$ is perpendicular to the surface at P. Therefore, since the horizontal component of \vec{F} is $m\omega^2 x$ and the vertical component is $-mg$, it follows that

$$(7) \quad f'(x) = -\frac{m\omega^2 x}{-mg} = \frac{\omega^2}{g} x .$$

If we integrate Equation (7), we obtain

$$(8) \quad y = f(x) = \frac{\omega^2 x^2}{2g} + C .$$

But $C = 0$ since $f(0) = 0$, so the function whose graph generates the surface of the liquid is

$$(9) \quad y = f(x) = \frac{\omega^2 x^2}{2g} .$$

Thus, the surface of the liquid is the surface of revolution generated by a parabola symmetric to the y-axis.

3. SOME ADDITIONAL COMMENTS

3.1 Parabolic Mirrors

Our main result suggests a method for making the parabolic mirrors that are used in reflecting telescopes: A plastic monomer is poured into a short cylindrical form and the form and monomer are rotated at a constant angular speed until the monomer is motionless relative to the form. A catalyst is then sprayed on the surface to polymerize the plastic so that the surface will "set". The parabolic surface of the plastic can then be aluminized to produce a parabolic mirror.

3.2 Planar Motions of a Mars Particle

Our derivation of the formula for the centripetal acceleration used the fact that the liquid particles follow a circular path at a constant angular speed

around the axis of rotation of the container. Calculus can also be used to study more complicated planar motions of a mass particle. You can find a good discussion of such motions in Section 12-5 of Calculus and Analytic Geometry - Fifth Edition by G. B. Thomas and R. L. Finney. Notice that Equation (11) in that section yields our formula for the magnitude of the centripetal acceleration when it is specialized to the case of rotation about a point at a constant speed. The more general analysis of planar motion is crucial to the study of planetary and satellite orbits.

4. REFERENCES FOR FURTHER READING

1. Polya, G. Mathematical Methods in Science Mathematical Association of America New Mathematical Library Series (1977). This book presents a very readable account of a variety of applications of mathematics including the problem considered in this pamphlet.
2. Swanson, W. M. Fluid Mechanics. New York. Holt, Rinehart and Winston (1970). The field of fluid mechanics makes heavy use of mathematics. This book provides an interesting and reasonably self-contained introduction to the subject.

5. EXERCISES

1. A cylindrical tank with a vertical central axis has a radius of 2 ft. and is partly filled with a liquid. Suppose that the tank and the liquid are rotating around the central axis at the same constant angular speed. The surface of the liquid is 5 ft. below the top of the tank at the central axis and 4 ft. below the top 1 foot out from the central axis.
 - (a) Compute the angular speed of rotation of the liquid.

- (b) How far below the top of the tank is the liquid surface on the interior face of the tank?
- (c) If the depth of the liquid is 2 ft. at the central axis, what is the volume of the liquid in the tank?
2. In our derivation of the formula for the magnitude of the centripetal acceleration, we approximated the length of the chord of a circle subtended by a small central angle $\Delta\theta$ by the length of the corresponding circular arc. Show that as $\Delta\theta$ approaches 0, the ratio of the lengths approaches 1. (Hint: Use L'Hopital's Rule or the fact that $\lim_{\theta \rightarrow 0} \frac{\sin\theta}{\theta} = 1$.)

6. ANSWERS TO EXERCISES

1a. 8 radians/sec.

1b. 1 foot.

1c. 16π ft.³