CONSORTIUM

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Special Edition: HiMCM Outstanding Papers

From the Editor's Desk

IN THE WORKS

GARY FROELICH

Welcome to our fourth HiMCM special issue. The HiMCM special moves to spring this year because the contest is now held in the fall. We were able to meet our goal of moving the special issue with the help of the contest director and judges who sent their articles to us only a few days after completion of the national judging, thereby meeting our copy deadline. We are especially grateful for the enthusiasm demonstrated by everyone involved in HiMCM—students, advisors, and judges. We urge all of our readers to examine the student work in this special issue. Outstanding work in HiMCM does not necessarily require relatively advanced mathematics. Creativity and the ability to communicate and defend results are very important. Students who do not do well in other mathematics contests can succeed in HiMCM. Check the COMAP web site, www.comap.com, for information on this fall's contest.

2002 promises to be an eventful year at COMAP. In the fall, Key College Press will publish our new developmental mathematics texts. We are in the process of revising *Mathematics: Modeling Our World* (M:MOW), our high school series, and expect to have Course 1 ready for field test by fall. We are also busy creating new high school modules in our TechMAP series. Our web site provides current information on the M:MOW revision and TechMAP, including how to become a field tester. ❑

CONSORTIUM

Consortium is a quarterly newsletter of the Consortium for Mathematics and Its Applications, Inc. (COMAP), but it is also much more. Each issue brings lessons and ideas that demonstrate what COMAP believes is an exciting way to teach and learn mathematics. The center Pull-Out Section of the issue is a classroom lesson ready to be photocopied and distributed to your students.

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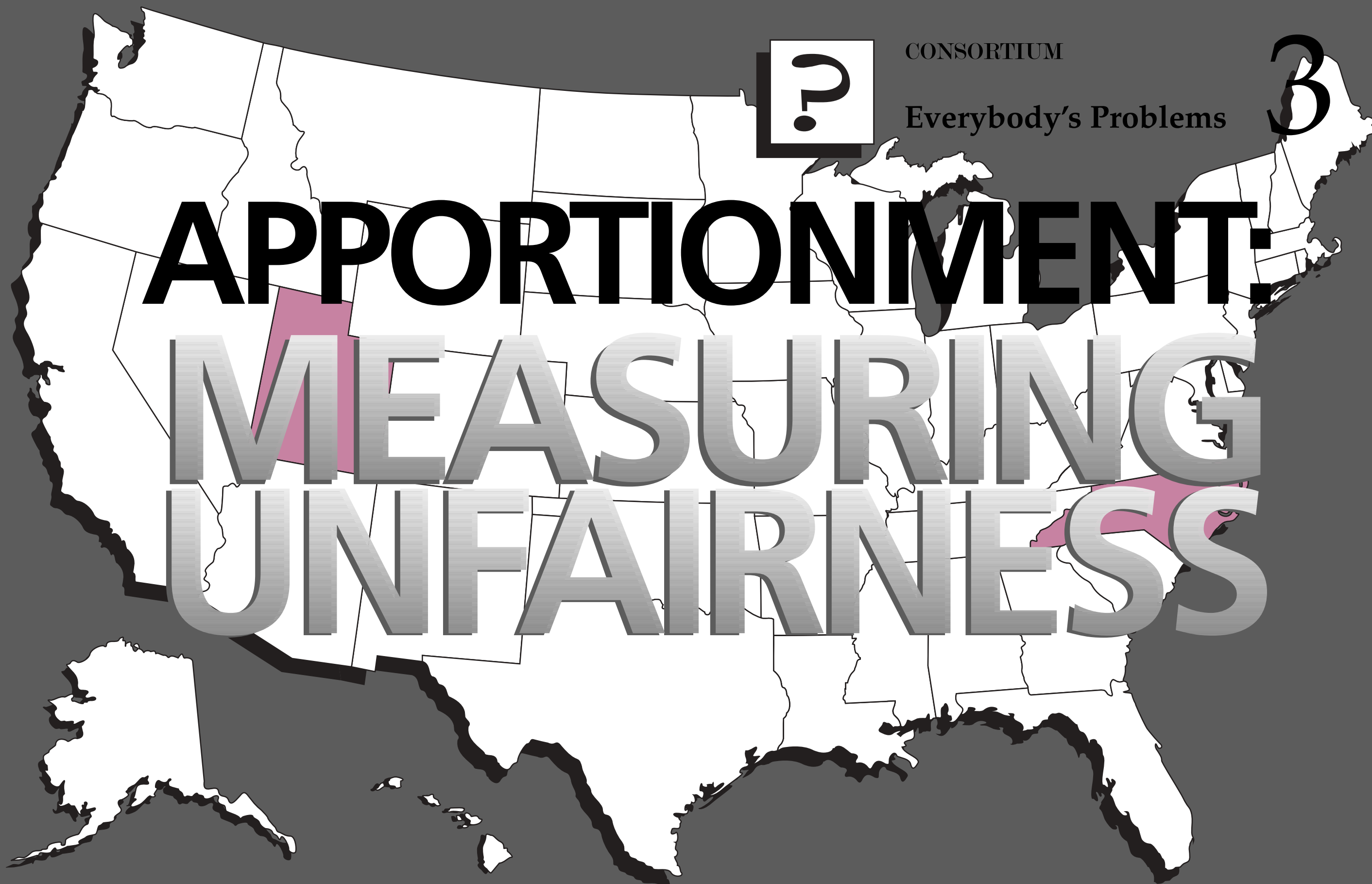
APPORTIONMENT: MEASURING UNFAIRNESS



DAN TEAGUE AND FLOYD BULLARD

Every ten years when the census is taken, the federal government rearranges or reapportions the delegates in the House of Representatives. This reapportionment is designed to give equal representation by assuring that all districts with the same population get equal numbers of representatives. After the 2000 census, the state of Utah sued the federal government arguing that it should be given the delegate that went to the state of North Carolina. What mathematics is behind the apportionment of the House of Representatives, and did Utah have a case?

Before we begin, let's consider developing our own methods of apportionment. Suppose there is a township with 5 districts and a total population of 20,000 as shown in **Table 1**.

District	A	B	C	D	E
Population	8150	5322	3188	2353	987

TABLE 1. POPULATION OF EACH DISTRICT.

If the governing body has 20 delegates, how should these delegates be apportioned to the districts? Give this problem to your students and have them apportion the 20 delegates as "fairly" as possible.

The Hamilton Solution

A common student solution develops as follows: With 20 delegates representing 20,000 people, there should be 1 delegate for each 1000 citizens. District A has 8150 citizens and therefore should be represented by 8.15 delegates. The number 8.15 is called the *quota* for district A. We will denote the quota for district A as $Q_A = 8.15$. Similar quotas can be found for the other districts. It is unfair if district A is given 9 delegates (or any number of delegates greater than 8.15 delegates), because district A would be overrepresented. However, it is also unfair if district A receives only 8 delegates, because then district A will be underrepresented. The different districts should get *at least* the integer part of their quota (shown in the third row of **Table 2**), that is district A should have at least 8 delegates, district B at least 5 delegates, district C at least 3 delegates, and district D at least 2 delegates. Additionally, since every district must be given representation, district E must get at least 1 delegate. However, the total of the integer parts is only 19. The essential question is which district should receive this last delegate?

District	A	B	C	D	E
Population	8150	5322	3188	2353	987
Quota (20 Delegates)	8.15	5.322	3.188	2.353	0.987
Quota (21 Delegates)	8.56	5.59	3.29	2.47	1.03

TABLE 2. POPULATION AND QUOTA FOR EACH DISTRICT.

The *Hamilton Method* gives the last delegate to the district with the largest fraction of a delegate in its quota. In this case, it goes to district D, which has 0.353 of a seat as the fractional part of its quota. So this apportionment plan would apportion 8 seats to A, 5 seats to B, 3 seats to C, 3 seats to D, and 1 seat to E. District B would be somewhat upset, most likely, since it missed getting the extra delegate by 32 people. This method of apportioning seats was first proposed by Alexander Hamilton and was used to apportion delegates to the House of Representatives of the U. S. Congress from 1850 to 1900. However, there is a problem with apportioning seats this way.

District B missed getting the extra delegate by 32 people. If we had 21 delegates, then district B might be appeased. Consider the same districts with the same populations, but with 21 seats to apportion. With 20,000 people and 21 delegates, then there should be 1 delegate for every 952 people. To determine the quota for each district, divide the population by 952.

If every district receives at least the integer part of their quota, then district A should have at least 8 seats, district B at least 5 seats, district C at least 3 seats, district D at least 2 seats, and district E at least 1 seat. Notice that the total is again only 19 seats. If we apportion the remaining 2 seats to the districts with the largest fractional parts, then districts B and A get the 2 unassigned delegates since their fractional parts are 0.59 and 0.56, respectively. So the apportionment plan devised by Hamilton would give district A 9 delegates, district B 6 delegates, district C 3 delegates, district D 2 delegates, and district E 1 delegate. District B is satisfied, but what happened to district D? When there were 20 delegates, district D was apportioned 3 representatives, but with 21 delegates, it gets only 2. When a seat is added, we expect that some districts will have an increase in their representation, but we do not expect to find that increasing the number of delegates would causes a district to lose representation!

The Hamiltonian Method seems a perfectly reasonable way to apportion delegates. Unfortunately, it leads to unacceptable results. A district may lose representation simply by having the total delegation increase in size. This is called the *Alabama Paradox*, since Alabama would have lost representation in 1880 if the size of the House of Representatives was increased to 300, the original goal. Instead they increased the total number of Representatives to 325 to avoid the surprising effect.

The Method of Differences: Apportioning on the Basis of an "Unfairness Index"

One definition of a fair apportionment procedure is one in which the same number of people in each state is represented by 1 delegate. If 1 delegate is given for every 500 people in one state then 1 delegate should be given for every 500 people in all other states. Any deviation from this would be unfair to some state.

Consider two states, X and Y, with 100,000 and 60,000 people, respectively. Suppose state X has 5 representatives while state Y has 4. Is this fair apportionment? State X has 1 representative for every 20,000 citizens while State Y has 1 representative for every 15,000 people. This is unfair to state X by 5000 people/delegate.

If state X has a population P_X and is represented by R_X delegates while state Y has a population P_Y and is represented by R_Y delegates, then the unfairness of this apportionment can be defined by the difference $\frac{P_X}{R_X} - \frac{P_Y}{R_Y}$.

If $\frac{P_X}{R_X} - \frac{P_Y}{R_Y} > 0$, the apportionment is unfair to X, and we call the value of $\frac{P_X}{R_X} - \frac{P_Y}{R_Y}$ the *unfairness index* to X.

If $\frac{P_X}{R_X} - \frac{P_Y}{R_Y}$ measures the unfairness to X, if state X has R_X delegates and state Y has R_Y delegates, then $\frac{P_X}{R_X} - \frac{P_Y}{R_Y + 1}$

measures how unfair it would be to X if Y got one additional delegate. Likewise, $\frac{P_Y}{R_Y}-\frac{P_X}{R_X+1}$ measures how unfair it would be to Y if state X got an additional delegate. These two expressions are mathematical models of unfairness. We can use them to develop an apportionment plan that will minimize the unfairness.

The conditions for which X will get a delegate rather than Y is that the measure of unfairness to Y when X gets the delegate is smaller than the measure of unfairness to X when Y gets the delegate. Symbolically, that means that X gets the delegate if

$$\frac{P_X}{R_X}-\frac{P_Y}{R_Y+1}>\frac{P_Y}{R_Y}-\frac{P_X}{R_X+1}.$$

If we rewrite the inequality in our statement by moving all terms with X to the left of the inequality and all terms with Y to the right side, then we can say "Give the delegate to X if $\frac{P_X}{R_X}+\frac{P_X}{R_X+1}>\frac{P_Y}{R_Y}+\frac{P_Y}{R_Y+1}$."

This inequality can be further simplified to "Give the delegate to X if $\frac{P_X(2R_X+1)}{R_X(R_X+1)}>\frac{P_Y(2R_Y+1)}{R_Y(R_Y+1)}$."

To determine which state gets the extra delegate, simply evaluate the expressions on either side of the inequality above and give the delegate to state X with the larger value. Moreover, the transitive property of inequalities allows us to include other states. If we have states A, B, C, D, and E, we compare the values of the expressions $\frac{P_A(2R_A+1)}{R_A(R_A+1)}$, $\frac{P_B(2R_B+1)}{R_B(R_B+1)}$, $\frac{P_C(2R_C+1)}{R_C(R_C+1)}$, $\frac{P_D(2R_D+1)}{R_D(R_D+1)}$, and $\frac{P_E(2R_E+1)}{R_E(R_E+1)}$ and give the delegate in question to the district with the largest value.

If we call the expression $\frac{P(2R+1)}{R(R+1)}$ the *U-value* (for unfairness), we should find the U-values for each district and give the extra seat to the state with the largest U-value. If we want to apportion 20 delegates to 5 districts, we give each district 1 delegate and compute the values of $\frac{P(2R+1)}{R(R+1)}$ as R varies from 1 to 10 (no district will receive more than 10 delegates) for each of the five populations. We then assign additional delegates to the 15 largest U-values. The table below gives each of these values for the five districts. The values in the table have been rounded to the nearest integer to facilitate the reading.

Notice that the largest number is 12,225, so the first new delegate goes to district A. The second largest is 7983, so the next delegate goes to district B. In **Table 3** we indicate the first 16 assignments in the order in which they were assigned. Again, compare these tables with the work previously done.

R	u_A	u_B	u_C	u_D	u_E
1	12,225 (1)	7983 (2)	4782 (4)	3530 (8)	1481
2	6792 (3)	4435 (6)	2657 (11)	1961 (15)	823
3	4754 (5)	3105 (9)	1860	1373	576
4	3668 (7)	2395 (13)	1435	1059	444
5	2988 (10)	1951	1169	863	362
6	2523 (12)	1647	987	728	306
7	2183 (14)	1426	854	630	264
8	1924	1257	753	556	233
9	1721	1124	673	497	208
10	1556	1016	609	449	188

TABLE 3. U-VALUES FOR R = 1 TO 10 WITH ASSIGNMENTS.

The final apportionment of 20 delegates by this scheme is district A with 8 delegates, district B with 5 delegates, district C with 3 delegates, district D with 3 delegates, and district E with 1 delegate. This is the same distribution as the Hamilton Method. Also notice that if the number of delegates was increased or decreased, we would quickly be able to apportion the delegates. The apportionment of 21 delegates would have given district B 6 delegates. Also notice that district B would receive 7 delegates before E gets its second, but E would get its second delegate before district D receives its third delegate.

Method of Equal Proportions

Our first method of apportionment was based on the principle that if two numbers are equal, their difference is zero. If $\frac{P_X}{R_X}=\frac{P_Y}{R_Y}$, then $\frac{P_X}{R_X}-\frac{P_Y}{R_Y}=0$ and the apportionment between X and Y is fair. The farther the difference $\frac{P_X}{R_X}-\frac{P_Y}{R_Y}$ is from zero, the less fair the apportionment. Another way to compare values is to consider their ratio. If two numbers are equal, their ratio is one. If $\frac{P_X}{R_X}=\frac{P_Y}{R_Y}$, then $\left(\frac{P_X}{R_X}\right)\Big/\left(\frac{P_Y}{R_Y}\right)=1$. The larger the ratio $\left(\frac{P_X}{R_X}\right)\Big/\left(\frac{P_Y}{R_Y}\right)>1$, the less fair the apportionment is to X. Let's see how this measure of fairness compares to the previous measure. Using the ratio $\left(\frac{P_X}{R_X}\right)\Big/\left(\frac{P_Y}{R_Y}\right)$ as our measure of unfairness to X, we repeat the argument from before. Give the disputed delegate to Y and compute the unfairness to X, then give the delegate to X and compute how unfair it is to Y. Give the measure to the district that will produce the smallest computed unfairness.

In this case, X will get the extra delegate when

$$\frac{\left(\frac{P_Y}{R_Y}\right)}{\left(\frac{P_X}{R_X+1}\right)}<\frac{\left(\frac{P_X}{R_X}\right)}{\left(\frac{P_Y}{R_Y+1}\right)}.$$

Simplifying the inequality, we have "Give the extra delegate to X if $\frac{(P_X)^2}{R_X(R_X+1)}>\frac{(P_Y)^2}{R_Y(R_Y+1)}$."

So district X is assigned the disputed delegate if $\frac{P_X}{\sqrt{R_X(R_X+1)}} > \frac{P_Y}{\sqrt{R_Y(R_Y+1)}}$. Notice that this is totally different from the previous measure and will apportion the delegates differently as well.

If we compute the unfairness values (denoted U') for each district with the rule $U' = \frac{P}{\sqrt{R(R+1)}}$, we generate **Table 4**.

R	U'_A	U'_B	U'_C	U'_D	U'_E
1	5763 (1)	3763 (2)	2254 (5)	1664 (8)	698
2	3327 (3)	2173 (6)	1301 (11)	961	403
3	2353 (4)	1536 (9)	920	679	285
4	1822 (7)	1190 (13)	713	526	221
5	1488 (10)	972 (15)	582	430	180
6	1258 (12)	821	492	363	152
7	1089 (14)	711	426	314	132
8	960	627	376	277	116
9	859	561	336	248	104
10	777	507	304	224	94

TABLE 4. U′-VALUES USING RATIO METHOD FOR R = 1 TO 10.

If the total representation is 20, then the delegates will be split so that district A has 8, district B has 6, district C has 3, district D has 2, and district E has 1. This is *not* the same representation as the Method of Differences produced. If 21 delegates are used, then the two apportionments agree.

This Equal Proportions method of apportionment is known as *the Huntington-Hill Method*, and is presently used to apportion the House of Representatives. The House now has a fixed number of seats, 435, that are apportioned to the states based on their population. In the last census, Utah had a population of 2,236,714 and North Carolina a population of 8,067,673. Computing the unfairness indices for each using $U'_U(R) = \frac{2,236,714}{\sqrt{R(R+1)}}$ and $U'_{NC}(R) = \frac{8,067,673}{\sqrt{R(R+1)}}$ we create **Table 5**.

R	Utah	NC
1	1,581,596	5,704,706
2	913,135	3,293,613
3	**645,684**	2,328,937
4	500,144	1,803,987
5	408,366	1,472,949
⋮	⋮	⋮
11	194,681	702,200
12	179,080	**645,931**
13	165,796	598,016

TABLE 5. U′- VALUES FOR UTAH AND NORTH CAROLINA.

North Carolina's Unfairness Index of 645,931 for $R = 12$ gave North Carolina the 435th seat in the house. It edged out Utah's Index of 645,684 for the last seat. If Utah's population had been 857 larger, then Utah's score of $U'_U(3) = \frac{2,237,571}{\sqrt{3(3+1)}} = 645,931.1$ would have been larger than North Carolina's score of $U'_{NC}(12) = \frac{8,067,673}{\sqrt{12(12+1)}} = 645,930.8$, and Utah would have received a 4th delegate instead of North Carolina receiving its 13th. Utah filed suit claiming that it had many more than 857 citizens of Utah overseas on Mormon missions and they were uncounted in the census. The suit was resolved in North Carolina's favor, but the Supreme Court has agreed to hear the case this summer.

Methods of apportionment offer both an interesting mathematical study and a rich history for students to investigate. The very first presidential veto was Washington's veto of the Hamiltonian apportionment in favor of one proposed by Thomas Jefferson. Daniel Webster proposed a Method of Differences using the measure of delegates per person $\frac{R_X}{P_X} - \frac{R_Y}{P_Y}$ rather than people per delegate. This leads to an entirely different apportionment. The references below present some of the historical and mathematical aspects of this intriguing subject. ❑

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Everybody's Problems concerns teaching high school mathematics courses with real-world problems, particularly problems that are suitable for students at all levels.

HUMOROUS HISTORICAL HIGHLIGHTS

RICHARD L. FRANCIS

There's very little to smile about in algebra class. Or geometry, trigonometry, and whatever else may align with that ultra-serious discipline called mathematics. Or is there? Humorous moments may trigger insights that lead into the more formalized aspect of the classroom.

History certainly has its lighter moments, whether they are in antiquity (as in the Archimedean bathtub account) or the modern era (as in the Newtonian apple episode). Or, if not that, then the speculated forms of mathematics in pre-history and its Stone Age antics. Many entertaining encounters appear as the pages of the history of mathematics are examined. Some relate to name mix-ups and some concern a ridiculous use of symbolism. Then there are the lighter moments of the more recent setting: glaring exaggeration, twisted logic, or even the legendary. Whether in narrative form, a poetic setting with a clever rhyming scheme, or in a cartoon format, each holds mathematics up in a positive manner that hopefully invites consideration of its more serious features.

The broader area of great events in world history has long proved a fertile field for the singling out of humor. Yet the narrow field of mathematics shares in this phenomenon and occasionally parallels more general or non-mathematical happenings. Perhaps it is helpful to set aside for a few minutes the somber outlook by glancing briefly at a few morsels of mathematical madness.

RUSSELL'S REACTION

Was Bertrand Russell (1872–1970) identifying with T. C. MITS (the celebrated man in the street) when he described mathematics as that area of study in which one never knows what he is talking about? This may well be the initial impression of some, especially as the role of undefined terms, primitive notions, assorted (totally unproved) axioms, and wildly chosen postulates are examined. Any attempt to define everything leads to weird outcomes, such as the need for an infinite vocabulary or some kind of circularity. Note, say, the predicament of defining the word "point" as "that which has no part but position only." It hardly suffices to assert "a part is that which a point hath not." Note too the predicament of proceeding with the word "point" regarded as an undefinable term. What meaning can thus be given to the declaration that "a line is a set of points" or "two points determine a line"? Is Russell trying to tell us something? Especially as the mathematician concludes that he really doesn't know what these rock-ribbed statements of geometry mean.

Rather than reply apologetically, perhaps the geometer or topologist or algebraist may point to such observations with pride. Most would be offended by the accusation "you don't know what you're talking about." However, the mathematician may glory in it and thus be highly complimented.

Many are the reactions to Russell's oft-quoted remark that "mathematics may be defined as the subject in which we never know what we are talking about nor whether what we are saying is true." What a blockbuster of a statement! Paradox, predicament, and profundity surround this guiding rule of conduct, a suitable motto for framing in any mathematician's office.

PERCEPTIONS

Are mathematicians truly absent-minded? Is it the case as well that great mathematicians cannot add a column of figures with any degree of confidence? Or that the professor of mathematics will write the letter *a* on the blackboard, call it *b*, mean *c*, but it really should be *d*? Stories abound in this area of common belief, some of which are handed down from ancient history.

Archimedes, the noted mathematician from Syracuse, is remembered for many discoveries. Yet he is perceived as one so involved in his work that outside, mundane, and everyday circumstances hardly mattered. To him the closing symbol QED may have meant "quite easily *did*" rather than "*quit* and *eat dinner*."

The story is told and re-told of his resolution of the problem of determining whether or not King Hieron's crown was of pure gold. To find the volume and hence the density of the crown was the true problem. However inclined toward cleanliness Archimedes was, the solution occurred as a splash flash of insight, better known as a Bathtub Theorem. In his unclothed excitement, he broadcast to his small corner of the world the word "Eureka," and its implication of water displacement as the method for finding the crown's volume. Though his report to King Heiron was disappointing, other more favorable achievements were to follow. So intensely committed to mathematics was he that his definition of "plain" geometry was figures drawn in the sand. It also occasioned the end of his career.

Archimedes, much like those before and after him, dreamed the dreams of mathematics. Be it Descartes' dreamland geometry, Newton's falling apple puzzler clearly denoting the "gravity of the situation," or Princess Dido's bull-hide approach to the isoperimetric problem, an element of the humorous also surfaces. Such are the awkward but funny circumstances under which many great discoveries are made.

False perceptions often take the form of humorous criticism. Newton himself was not spared as stories abound concerning his absent-mindedness. But his greatest achievement makes an appearance as well. Note the words of the poet:

Hear now bishop Berkeley exclaim,
As Newton he questions by name.
Departed almost,
Each quantity's ghost,
Your fluxion's a logical shame.

In spite of the appropriate description of his "going off on a tangent," the advent of the calculus splits mathematical history in half.

No doubt, all have heard of the great writer of children's books named Lewis Carroll. Though his real name was Charles L. Dodgson (1832–1898), his famous works called *Through the Looking Glass* and *Alice's Adventures in Wonderland* made a hit with the English speaking world of the nineteenth century. Such popularity continues today.

However, not all of his writings were stories for children. Some publications were mathematical textbooks, an activity fully in line with his position of lecturer in Oxford.

The story is related that Queen Victoria (1819–1901) was so impressed with *Alice's Adventures in Wonderland* that she requested a copy of Carroll's next book. Hardly did she know that this forthcoming book was a theory of parallels, not exactly what was expected. Other publications of Carroll proved just as mysterious and unreadable to Queen Victoria as the one on mathematics.

Carroll's activities with children's groups were to take other humorous and entertaining forms. Yet one can barely imagine the queen's amazement in opening the pages of a geometric treatise while fully expecting a more leisurely, enjoyable perusal of a story for children.

Humorous perceptions of mathematicians are many and extend from Thales, Pythagoras, and Archimedes of antiquity to Galileo, Newton, and Descartes of the early modern period. Quite likely, the present shares in this recurring pattern of light-hearted perceptions of "who's who" and "what's going on" in mathematics.

MOTHER GOOSE MATHEMATICS

Children's viewpoints may extend to theorems, derivations, formulas, and varied encounters on the mathematical landscape. Such appear in the poet's colorful touch on finding detailed approximations to roots of algebraic equations. That is,

William G. Horner, sat in a corner,
Finding his surds a new way.
By a choice rule of thumb,
He figured out some,
It's a method we all use today.

Or, if not that, then the Quotient Rule for finding the derivative "dee," by using the numerator "hi" and the denominator "lo."

Lo dee hi, Less hi dee lo;
And over lo by lo.
In Newton's school,
this quotient rule,
Is very nice to know.

Not even the unsolved problems of today's mathematics are overlooked, as in

It appears to work out every time,
This conjecture of Goldbach sublime,
Numbers starting with four,
Even so, evermore,
Decompose to a prime plus a prime.

Or silly little verses that repeat well known facts from arithmetic or even advanced mathematics, such as

Now Euler decided to try,
Find e to the power xi,
"Add a one, take the sum,
The result will become,
A zero, if x equals pi."

And the poetry continues. Yet each symbolizes in its own way some great mathematical happening. As noted, the few here listed range over the immense contributions to the theory of equations by the Englishman Horner, the formalizing of the calculus as in its earlier years, the abundant listing of unsolved problems in classical number theory, and Euler's eighteenth century delvings into the world of complex numbers. These theoretical advances, though pictured in an amusing setting, all symbolize repeated stories of mathematical application.

MUCH ADO ABOUT NOTHING

Marin Mersenne, the noted French friar of number theory fame, contributed immensely to the study of even perfect numbers. Others in time were to consider the mathematical counterpart in a parity pursuit. In this quest for odd perfect numbers, such impressive names as Euler and Sylvester appear. Impressive also is the voluminous writings of these and others concerning the properties of odd perfect numbers. What strikes the onlooker as amusing are the stacks and stacks of pages written on the subject of something that may not exist. All in all, we may be examining a treatise on the empty set. The existence of a treatise on the subject of the (possibly) non-existent could well be an excessive and excited description of nothing.

A trend in modern mathematics is that of "more and more about less and less" (as in celestial hydrostatic differential distance geometry, whatever that may be). Perhaps the odd perfect number quest jokingly suggests the indeterminate form of knowing everything about nothing.

To use the amusing terminology of Edward Kasner, it is known that no odd perfect number can be found less than a googol. Not even a googol squared or a googol cubed. The googolplex may however provide a different outcome. The understated description in this quest is the simple exclamation; "these numbers are getting big."

One should remember that searches for elusive numbers have sometimes met with success. A persistent search for non-algebraic numbers by Euler and others of the eighteenth century proved fruitless. But time made the difference as evidenced by Liouville's discovery of the first known transcendental. A conceivable nothingness was then and there replaced by the infinite. His driving slogan may well have been, "if I could find just one!" Today, the dam has burst as the set of transcendentals is known to be not only infinite but also uncountable.

In the murky realm of nothingness, such terminology including zero, the empty set, and vacuous concepts, somehow surfaces. Fallacies or surprising outcomes stemming from "nothing notions and notation" are often met with a smile.

BESIEGED MATHEMATICS

Mathematics today abounds in unsolved problems. History is also replete with problem after monumental problem ultimately solved or shown impossible. None better illustrates this than the "attack on antiquity" called "The Big Three."

These famous problems of ancient Greek geometry have plagued zealous mathematicians across the centuries and have brought would-be solvers to the place of more meaningful description. Thus the angle trisection challenge may be fittingly called "Anglis Trisectis Tiringus," a daring leap upward from "bisectis." This is fittingly accompanied by the well-known "cubum dupliphobia" and "quadrocirclo obsessicus." Fortunately, the plague abated in the nineteenth century, due to the helpful hands of Pierre Laurent Wantzel and C. Ferdinand Lindemann.

Though of lesser duration, the famous challenges of the Four-Color Map Problem and Fermat's Theorem come to mind. Both problems were resolved only lately. Paraphrasing Fermat's classic note and in connection with some other great problem, one might encounter the hastily scrawled note on the bus station wall that reads, "I have discovered a truly remarkable proof of the Riemann Conjecture, but unfortunately, the bus has arrived and I do not have time to write it." Perhaps, upon discovery of the note, the great conjecture could thus be called a theorem, and accordingly lead to a three hundred-year quest for its tantalizing proof.

PRINCIPLE, PARADOX, PARAPHRASE

The humorous is ubiquitous as the timeline of mathematics is traced. Note such random perceptions, perplexities, and platitudes as

The Lazy Algebraist's Principle. If two x's appear on the same piece of paper, they may be cancelled.

Dedekind's Parachute Paradox. Where was the man when he jumped from the plane? Not on the plane as that was before he jumped. And not in the air as that was after he jumped. So where was he?

Lumber, Longevity, and LaPlace. These logs will double the life of the astronomer.

Cardan to Tartaglia. More lies ahead.

And on and on! Indeed more lies ahead as one chooses to scan the historical record, a record brightened in some measure by its fascinating blend of application, theory, and the occasional smile. ❑

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Richard L. Francis (ed.) is a professor of mathematics at Southeast Missouri State University, where he has taught since 1965. His major scholarly interests include number theory of the history of mathematics.

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IN "PLANE" VIEW

PETER ALLISON, LOUIS BASEL,
OANA CIORBEA,
SHANTANU DHAKA,
BROOKS GORDON,
REBECCA HUDSON,
BETSY LEE, DEREK LEE,
JOHN PLAYFORTH,
MICHAEL XIAO

Introduction by: Jon Choate

This article was produced as an assignment for a computer graphics course that I am currently teaching to high school seniors. The assignment was to produce an image of a cube in perspective with hidden lines removed. The article describes the mathematics the students used to produce the image. They produced the actual image using Microsoft Excel. If any of you would like a copy of the spreadsheet, send your email address to me at jchoate@groton.org and I will email it to you. We are also in the process of putting up a website dedicated to applications of secondary mathematics in computer graphics. If you have any material you would like to contribute to this site, please contact me and I would be glad to add it to the site if it is suitable. There will be more about the site in the next edition of *Consortium*. ❑

In the world of computer graphics, the essential goal is to portray a three-dimensional world on a two-dimensional surface. The task presented to programmers is to accomplish this goal in the most efficient way possible. As a class, we tackled this problem in the context of a cube. In order to portray the cube on a two-dimensional computer screen, we set up a projection scheme that maps three-dimensional points onto a single plane while retaining perspective. Essentially, we needed a point to serve as an "eye," and a plane onto which all points will be mapped. In our case, we assigned the xy plane to be our projection plane and the point $(x_{eye}, y_{eye}, z_{eye})$ to be our eye.

To properly project our cube, we first needed to find the projected image of each of the vertices. To find these projected vertices, we connected each preimage point to the eye. The intersection of that line and the xy plane was the projected point. We let $(x_{point}, y_{point}, z_{point})$ be a point in three-dimensional space on the opposite side of the xy plane as the eye $(x_{eye}, y_{eye}, z_{eye})$ as seen in **Figure 1**, and we derived the formulas for its projection as follows.

The parametric equation of the line between the eye and our point is $(x, y, z) = (1 - t) * (x_{eye}, y_{eye}, z_{eye}) + (t) * (x_{point}, y_{point}, z_{point})$. Our projected point was on this line; specifically, it existed where $z = 0$. To solve for t when $z = 0$, we solved the separate z equation, $z = (1 - t) * z_{eye} + (t) * (z_{point})$. After some algebra, this left us with the equation $t = (-z_{eye})/(z_{point} - z_{eye})$. Once we calculated t, we inserted that value back into our original equation and found the equation for the x and y coordinates of the projected image. The equations we found were $x = (1 - t) * x_{eye} + t * x_{point}$ and $y = (1 - t) * y_{eye} + t * y_{point}$. To simplify our calculations we placed our eye on the z-axis so that $x_{eye} = y_{eye} = 0$, and our final equations for the projected points were $x = (-z_{eye})/(z_{point} - z_{eye}) * x_{point}$ and $y = (-z_{eye})/(z_{point} - z_{eye}) * y_{point}$. We used these equations to calculate the projected image for a vertex of our cube, and we repeated this for each of the subsequent vertices. We plotted these points on a graph, and connected the respective vertices to create a wire-frame image of the cube.

However, a wire-frame is often not the way in which a programmer wishes to display a three-dimensional object. When looking at a solid cube we see full faces, not a wire-frame, and it is possible to see a maximum of three faces as the others are hidden behind the visible faces. To account for this in our projection, we needed to find a method for identifying hidden faces. The only faces of an object that we can see are those that are tilted towards our eye. In other words, a face is visible only if the vector perpendicular to the face (the normal vector) forms an angle less than 90 degrees with a vector from a point on the face to the eye. For example, in **Figure 2**, the vector from the eye to the face BFGC forms an angle greater than 90 degrees with the normal to this face, so this face is not visible. Conversely, the vector from the eye to the face AEHD has an angle of less than 90 degrees with the normal to this face, so this face is visible. The dot product serves as a convenient test to see if the angle between our two vectors is greater or less than 90 degrees. If the dot product of these two vectors is positive, the angle is less than 90 degrees and the face is visible. Conversely, if the dot product is negative or zero then the face is hidden.

To calculate the normal vector, we took the cross product of two edges of each face (Edge 1 x Edge 2 = Normal). In order to ensure that the normal vector was directed away from the center of the cube, we made use of the right hand rule.

Using the normal vector of each face, we took the dot product of the normal vector and the vector from a point on that face to the eye. Using our dot product test, we determined if the face was visible. All visible faces were plotted and filled in, while the faces with non-positive dot products were not displayed. We were thus able to create an image of the cube that is both accurate in terms of perspective and realistic in its display of solid faces.

Having created our projection scheme, we wanted, finally, to develop a test that showed that it maintained perspective. The test that we created is as follows: we imagined that we placed parallel lines within our three-dimensional space and then found their projected images using our scheme. The projection of two parallel lines onto a two-dimensional viewing plane using our projection scheme should result in the convergence of the lines at a vanishing point. We used our viewing plane, which consisted of the plane $z = 0$ (the xy-plane) and points of the form $(x, y, 0)$. Two lines are parallel if they have the same directional vectors. Therefore, the set of lines, $(x, y, z) = (x_1, y_1, z_1) + r(a, b, c)$ are parallel regardless of x_1, y_1, and z_1 if a, b, and c are constant. Thus the projection of the point (x, y, z) as r goes to infinity should be the same no matter what initial point (x_1, y_1, z_1) is chosen. To demonstrate that the projections are independent of (x_1, y_1, z_1) as r goes to infinity and thus show that all parallel lines go to the same vanishing point, we first considered x' and y', which we assumed to be the x and y projections of a point along the line $(x, y, z) = (x_1, y_1, z_1) + r(a, b, c)$ for any r.

$$x' = \frac{z_{eye} * (x_1 + ar)}{-z_{eye} + (z_1 + cr)}$$

$$y' = \frac{z_{eye} * (y_1 + br)}{-z_{eye} + (z_1 + cr)}$$

Here, we created two vector form equations based on the projection scheme detailed above that we derived from the point form equations:

$(x', y') = (t * x_1, t * y_1)$, with

$$t = \frac{-z_{eye}}{z - z_{eye}}.$$

We then took the limit as r goes to infinity as a way to find the projection of the vanishing point.

$$\lim_{r \to \infty} \frac{z_{eye} * (x_1 + ar)}{-z_{eye} + (z_1 + cr)} = \frac{-z_{eye} * a}{c}$$

$$\lim_{r \to \infty} \frac{z_{eye} * (y_1 + br)}{-z_{eye} + (z_1 + cr)} = \frac{-z_{eye} * b}{c}$$

The constant $-z_{eye}$ in the denominator and the constants x_1, y_1, and z_1 of each equation all become negligible as r approaches infinity. The equation for the projected vanishing point of any line $(x, y, z) = (x_1, y_1, z_1) + r(a, b, c)$ is thus:

$$(x, y) = \left(\frac{z_{eye} * a}{c}, \frac{z_{eye} * b}{c}\right).$$

We therefore found, according to the equation above, that only the direction of the vector (a, b, c) and the initial placement of the eye z_{eye} affect the location of the vanishing point. Thus, two lines of the form $(x, y, z) = (x_1, y_1, z_1) + r(a, b, c)$ have the same vanishing point so long as a, b, and c are the same for both lines. In Figure 3, we show how parallel lines, such as DH and CG, have a common vanishing point in our projection scheme as perspective demands. Thus, our test of our projection scheme was successful. ❑

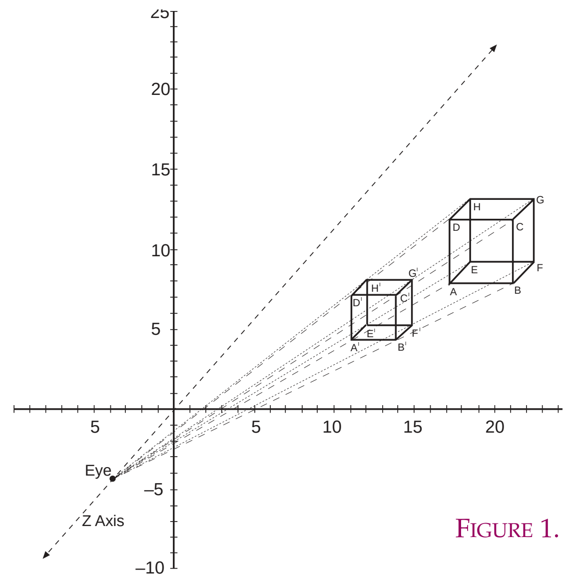

FIGURE 1.

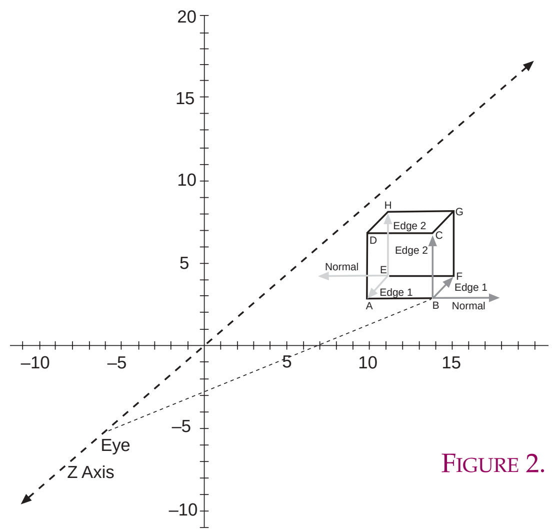

FIGURE 2.

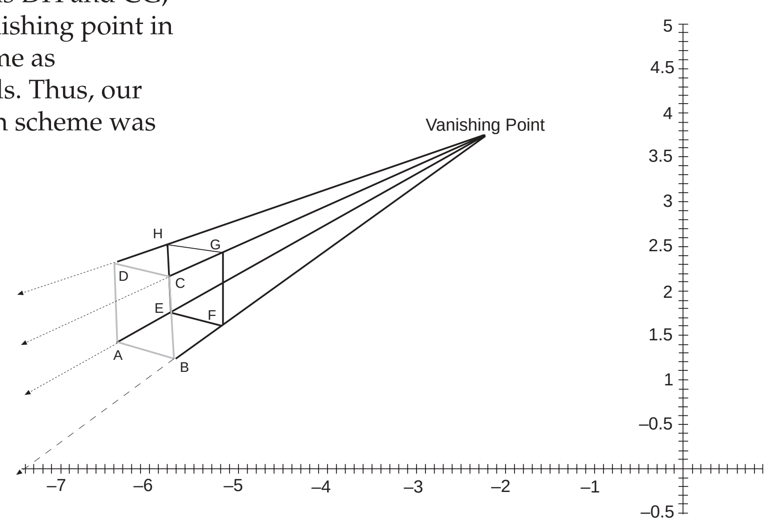

FIGURE 3.

Send solutions to old problems and any new ideas to the Modeler's Corner editor: Jonathan Choate, Groton School, Box 991, Groton, MA 01450.

CONSORTIUM
Talent Search

U.S.A. Mathematical Talent Search

ROUND 4

BLAIR KELLY

COMMENTS/HINTS/ANSWERS

Round 3—Year 13—Academic Year 2001–2002

1/3/13 The answer to this problem is 84. Our Problem Editor, Professor George Berzsenyi, suggested this problem.

2/3/13 Again, we are thankful to our Problem Editor for proposing this nice problem. The abundance of such triples is partially due to the many divisors of 84.

3/3/13 This problem was created by Professor Bruce Reznick of the University of Illinois and was first featured in the 1990 Indiana College Mathematics Competition. We are most thankful to Professor Reznick for his contribution.

4/3/13 An article about M.C. Escher's work in the April 1998 issue of *Pythagoras*, and excellent journal for high school students in the Netherlands, inspired Professor Berzsenyi to propose this problem. There are two points of triple intersection within the triangle. Escher divided the sides of the triangle into 3, 4, and 5 parts and discovered that there are 17 such triple points of intersection within the triangle.

5/3/13 This intersecting geometry problem was originally proposed for the American Invitational Mathematics Examination (AIME) by the late Professor Joseph Konhauser, a superb problemist and a great friend of Professor Berzsenyi. The sphere has a radius of 30 units.

****

Student's solutions to these problems will appear on the web site http://www.nsa.gov/programs/mepp/usamts.html

PROBLEMS

Round 4—Year 13—Academic Year 2001–2002

1/4/13 In a strange language there are only two letters, *a* and *b*, and it is postulated that the letter *a* is a word. Furthermore, all additional words are formed according to the following rules:

Given any word, a new word can be formed from it by adding *a b* at the right-hand end.

If in any word a sequence *aaa* appears, a new word can be formed by replacing the *aaa* by the letter *b*.

If in any word the sequence *bbb* appears, a new word can be formed by omitting *bbb*.

Given any word, a new word can be formed by writing down the sequence that constitutes the given word twice.

For example, by (D), *aa* is a word, and by (D) again, *aaaa* is a word. Hence by (B) *ba* is a word, and by (A) *bab* is also a word. Again, by (A), *babb* is a word, and so by (D), *babbabb* is also a word. Finally, by (C) we find that *baabb* is a word.

Prove that in this language *baabaabaa* is not a word.

2/4/13 Let $f(x) = x \cdot \lfloor x \cdot \lfloor x \cdot \lfloor x \rfloor \rfloor \rfloor$ for all positive real numbers x, where $\lfloor y \rfloor$ denotes the greatest integer less than or equal to y.

Determine x so that $f(x) = 2001$.

Prove that $f(x) = 2002$ has no solution.

3/4/13 Let f be a function defined on the set of all integers, and assume that it satisfies the following properties:

A. $f(0) \neq 0$;

B. $f(1) = 3$; and

C. $f(x)f(y) = f(x + y) + f(x - y)$ for all integers x and y.

Determine $f(7)$.

4/4/13 A certain company has a faulty telephone system that sometimes transposes a pair of adjacent digits when someone dials a three-digit extension. Hence a call to x318 would ring at either x318, x138, or x381, while a call received at x044 would be intended for either x404 or x044. Rather than replace the system, the company is adding a computer to deduce which dialed extensions are in error and revert those numbers to their correct form. They have to leave out several possible extensions for this to work. What is the greatest number of three-digit extensions the company can assign under this plan?

5/4/13 Determine the smallest number of squares into which one can dissect a 11 x 13 rectangle, and exhibit such a dissection. The squares need not be of different sizes, their bases should be integers, and they should not overlap.

****

Complete, well-written solutions to at least two of the problems above, accompanied by a **Cover Sheet**, should be mailed to

USA Mathematical Talent Search
DDM Co.
279 East Central Street, Suite 246
Franklin, MA 02038-1317

and **postmarked no later than 17 March 2002**. Each participant is expected to develop solutions without help from others.

*The USAMTS is a **free** high school mathematics competition designed to challenge talented students with difficult mathematical problems while providing a realistic time limit (at least a month). We wish to foster not only insight, ingenuity, and creativity, but also the virtue of perseverance, which is equally essential in scientific endeavors. Students may use any materials—books, calculators, computers—but all the work must be their own. Details are available at http://www.nsa.gov/programs/mepp/usamts.html.*

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Minty's lemma

Teachers using COMAP'S *Mathematics: Modeling Our World* will present workshops at the National Council of Teachers of Mathematics Annual Meeting in April.

"Mathematical Walks, Near-Collision Stunts, and Newton's Laws: Patterns in Motion"
Session 231 Workshop
April 22, 2002, 12:00 –1:30 P.M.
Harrah's Studio 1
Phyllis Kisch and Karen Hyers
Teachers, Tartan High School, Oakdale, MN

"Mathematical Modeling: The Glue that Binds"
Session 409 Workshop
April 22, 2002, 4:00 P.M. – 5:30 P.M.
Harrah's Studio 1
Rick Jennings
Teacher, West Valley High School, Yakima, WA

"Bring Hershey's Kisses®, M&M's® to a Higher Level—False Positives, Margins of Error, and Confidence Intervals"
Session 422 Minicourse
April 23, 2002, 8:00 A.M. – 11:00 A.M.
Room 103 Sands EXPO Center
Kay Shager, former teacher and Math Curriculum Chairperson; Peggy Sanders, teacher, North High School; and Judy Rohde teacher, John Glenn Middle School, North St. Paul, MN

HIGH SCHOOL MATHEMATICAL CONTEST IN MODELING

OUTSTANDING PAPERS

HiMCM

November 2001

The contest offers students the opportunity to compete in a team setting using applied mathematics in the solving of real-world problems.

Major funding provided by the National Science Foundation.

NSF Project Award Number ESI-9708171

Editor's Comments

This is our fourth HiMCM Special Issue. Since space does not permit printing all of the ten national outstanding papers, this special section includes the summaries from eight of the papers and edited versions of two. We emphasize that the selection of these two does not imply that they are superior to the other outstanding papers. They were chosen because they are representative and fairly short. They have received light editing, primarily for brevity. We also wish to emphasize that the papers were not written with publication in mind; the contest does not allow time to revise and polish. Given the 36-hour time limit, it is remarkable how well written many of the papers are.

We appreciate the outstanding work of students and advisors and the efforts of our contest directors and judges. Their dedication and commitment have made HiMCM a big success. We also wish to note that this special section takes the place of our regular HiMCM column, which debuted last year under the editorship of contest director Bill Fox. HiMCM Notes will return in the next issue.

Contest Director's Article

William P. Fox

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The High School Mathematical Contest in Modeling (HiMCM) completed its fourth year in excellent fashion. The growth of students, faculty advisors, and the contest judges is very evident in the professional submissions and work being done. The contest is still moving ahead, growing in a positive first derivative, and consistent with our positive experiences from previous HiMCM contests.

This year the contest consisted of 212 teams (a growth of about 30% from last year) from thirty-four states, with ten of these teams from outside the USA: one from Canada, four from the Hong Kong International School, and five from Department of Defense schools. Thus our contest continues to attract an international audience. The teams accomplished the vision of our founders by providing *unique* and *creative mathematical* solutions to complex open-ended real world problems. This year the students had a choice of two problems.

Problem A: Adolescent Pregnancy

You are working temporarily for the Department of Health and Environmental Control. The director is concerned about the issue of teenage pregnancy in the region. You have decided that your team will analyze the situation and determine if it is really a problem in this region. You gather the following 2000 data.

County	Age 10–14 Pregnant	Age 15–17 Pregnant	Age 18–19 Pregnant	10–14 births	15–17 births	18–19 births	10–14 births-unmarried	15–17 births-unmarried	18–19 births-unmarried
1	29	350	571	17	281	437	16	164	193
2	24	303	567	13	206	466	13	151	233
3	40	422	691	29	307	546	28	251	366
4	21	201	356	18	184	326	15	137	180
5	16	156	357	11	109	254	10	99	161
6	44	523	970	33	442	803	32	293	396
7	17	263	434	9	201	345	7	113	168
8	23	330	427	16	256	444	14	160	210
9	13	123	221	10	113	199	9	78	106
10	41	467	950	24	446	686	22	279	331
11	28	421	713	18	343	615	15	219	328
12	9	179	311	8	145	261	7	114	162

1998

Age	Pregnancies	Births
10–14	320	231
15–17	4041	3222
18–19	6387	5164

1999

Age	Pregnancies	Births
10–14	309	208
15–17	3882	3048
18–19	6714	5391

Build a mathematical model and use it to determine if there is a problem or not. Prepare an article to the newspaper that highlights your result in a novel mathematical relationship or comparison that will capture the attention of the youth.

Problem B: Skyscrapers

Skyscrapers vary in height, size (square footage), occupancy rates, and usage. They adorn the skyline of our major cities. But as we have seen several times in history, the height of the building might preclude escape during a catastrophe either human or natural (earthquake, tornado, hurricane, etc.). Let's consider the following scenario. A building (a skyscraper) needs to be evacuated. Power has been lost so the elevator banks are inoperative except for use by firefighters and rescue personnel with special keys. Build a mathematical model to clear the building within X minutes. Use this mathematical model to state the height of the building, maximum occupancy, and type of evacuation methods used. Solve your model for X = 15 minutes, 30 minutes, and 60 minutes.

COMMENDATION

All students and their advisors are congratulated for their varied and creative mathematical efforts. Of the 212 teams, 159 submitted solutions to the B problem and 53 submitted solutions to the A problem. The thirty-six continuous hours to work on the problem provided (in our opinion) a vast improvement in the quality of the papers. Teams are commended for the overall quality of work.

Again, many teams had female participation, showing this competition is for both male and female students. Teams again proved to the judges that they had "fun" with their chosen problems, demonstrating research initiative and creativity in their solutions. The fourth year effort was deemed a success!

JUDGING

We ran three regional sites in December 2001. Each site judged papers for both problems A and B. The papers judged at each regional site were not from their respective region. Papers were judged as Outstanding, Meritorious, Honorable Mention, and Successful Participant. All regional finalist papers for the Regional Outstanding award were brought to the National Judging. For example, eight papers may be discussed at a Regional Final and only four selected as Regional Outstanding, but all eight papers are brought and judged for the National Outstanding. Papers receive the higher of the two awards. The National Judging chooses the "*best of the best*" as National Outstanding. The National Judges commended the Regional Judges for their efforts and found the results were consistent. We feel that this regional structure provides a good prototype for the future of the contest's structure as it continues to grow.

JUDGING RESULTS:

National & Regional Combined Results

Problem	**National Outstanding***	Regional Outstanding	Meritorious
A	**1**	6	12
B	**9**	8	39
Total	**10**	14	51

Problem	Honorable Mention	Successful	Total
A	20	14	**53**
B	84	19	**159**
Total	104	33	**212**

GENERAL JUDGING COMMENTS

The judges' commentaries (written by Patrick Driscoll and Frank Giordano) provide specific comments on solutions to the specific problems. As contest director and head judge, I would like to speak generally about team solutions from a judge's point of view. Papers need to be coherent, concise, and clear. Students need to restate the problem in their own words so that the judges can determine the focus of the paper. Papers that explain the development of their model, assumptions, and its solutions and then support their findings mathematically generally do quite well. Modeling assumptions need to be listed and justified, but only those that come to bear on the team's solution (that can be part of simplifying the model). Laundry lists of assumptions that are never referred to in the context of the model development are not considered relevant and deter from the paper's quality. The model needs to be clearly developed and all variables that are used need to be defined. Thinking "outside of the box" is also considered important by judges. This varies from problem to problem but usually includes model extensions or sensitivity analysis of the solution to the teams' inputs. A clear conclusion and answers to specific scenario questions are all key components. The strengths and weakness is where the team can reflect on the solution and provide comments on its strengths and weaknesses. Attention to detail and proofreading the paper prior to final submission are also important as the judges look for clarity and style.

CONTEST FACTS

- A wide range of schools/teams competed, including teams from Canada, Portugal, Netherlands, and Hong Kong.
- 45%, or 95 of 212 teams, had female participation. 28 teams were all female.
- 55% teams were all male.
- There were 5 all female teams awarded National or Regional Outstanding.
- 35 states participated in the contest.

THE FUTURE:

The contest, which attempts to give the underrepresented an opportunity to compete and achieve success in mathematics endeavors, appears well on its way in meeting this important mission.

We continue to strive to grow. Again, any school/team will be allowed to enter the contest, as there will be no restrictions on the numbers of schools entering. A regional judging structure will be established based on the number of teams competing.

These are exciting times for our high school students. Mathematics continues to be more than learning skills and operations. Mathematics is a language that involves our daily lives. Applying the mathematical principles that one learns is the key to future success. The ability to recognize problems, formulate a mathematical model, solve, compute with technology, communicate, and reflect on one's work are keys to success. The ability to use technology aggressively to discover, experiment, analyze, resolve, and communicate results are also keys to success in the future. Students learn confidence by tackling ill-defined problems, and working together to generate a solution. Through team building and team effort solutions are built. Applying mathematics is a team sport.

Advisors need only be a motivator and facilitator. They should allow students to be *creative* and *imaginative*. It is not the technique used but the process that discovers how assumptions drive the techniques that are fundamental. Advisors should let students practice to be problem solvers. I encourage all high school mathematics faculty to get involved, encourage students, make mathematics relevant, and open the doors to success.

Mathematical modeling is an art and a science. Through modeling, students learn to think critically, communicate efficiently, and be confident, competent problem solvers for the new century.

CONTEST DATES

Mark your calendars. The next HiMCM will be held from 1–18 November 2002. Registrations are due by October 25, 2002. Papers must be postmarked by November 20 and mailed directly to COMAP. Teams will have a consecutive 36-hour block within this window to complete the problem. Teams can register via the worldwide web at www.comap.com.

HiMCM Judges Commentary

Problem A

Patrick Driscoll

Department of Mathematical Sciences
U. S. Military Academy
West Point, NY 10996

Perhaps the most striking element at this year's National Judging was the high quality of writing. Student teams are to be commended on their ability to effectively craft a technical report that both meets the requirements of the problem statement and is sensitive to the issue of readability. In years past, it was easy to identify those cases in which student teams distributed different tasks to members of their group and then simply merged the results of these efforts together and called it a report. With few exceptions, the papers we read at Nationals read with a single voice, indicating that teams are allocating a segment of time within their contest window to polishing their final product. Well done!

HiMCM judges enjoy reading team papers responding to problems that appear to be simply stated, but require considerable discussion on the part of teams before they go charging into solving the problem. The Adolescent Pregnancy problem is of this nature. The issue at hand is readily apparent, yet upon investigation subtleties arise that pose interesting open-ended challenges.

At the onset, teams had to first recognize that the question centered about a public policy issue in the hypothetical target region. For most of the papers, this became the basis for the newspaper article required in the problem statement. Teams universally rationalized that the Department of Health and Environmental Control is a governmental agency in the United States whose business it is to track behavioral trends, such as teenage pregnancy. The very first task teams had to accomplish was to define "the problem." How they decided to do this dictated the course of their modeling efforts. With minor exceptions, most teams answered this challenge by stating that a problem existed in the region if: (1) the yearly rate of teenage pregnancies was on a rise within the target region, independent of a national average; (2) the number of teenage pregnancies in the target region was at a higher level than that of a national average; or (3) both.

All papers dismissed the notion that *any* rate of occurrence other than that dictated by a "zero tolerance" policy for the age groups noted was a problem, particularly after performing Internet research at credible websites such as www.ganet.org, which provides statutory information concerning the age at which a person could be legally married in the State of Georgia:

> "Be at least 16 years of age. If either applicant is under the age of majority, parental consent shall be required, as provided in Code Section 19-3-37. However, the age limitations contained in this paragraph shall not apply upon proof of pregnancy on the part of the female or in instances in which both applicants are the parents of a living child born out of wedlock, in which case the parties may contract marriage regardless of age."

The problem statement made no mention of the marital state of the individuals represented in the data, nor to their financial state, nor to their education level, nor to whether or not double-counting could be occurring (i.e., if an 18 year old female was 6 months pregnant on January 1, 1998, delivered a healthy child in March, 1998 and became pregnant again in August 1998, was she counted once or twice in 1998?). Teams did an outstanding job of constructing appropriate assumptions to address these concerns and other subtleties associated with the data provided.

The better papers recognized a need to analyze the issue from *both* an intra-regional and an inter-regional perspective even though no inter-regional data was provided. An intra-regional perspective focuses on how the data has changed within the region as time progresses from 1998 to 2000. If a particular year group's rate of change is large but decreasing, the presumption is that the problem exists but is improving. If the rate of change is small but increasing, the problem is getting worse, and so on. An inter-regional perspective compares the data provided to regions outside of the target region, perhaps another state, county or to the country at large. This second perspective places the target region's data in a broader context, or in "the big picture." What may appear to be a problem within the region may not be at a level sufficient to dictate concern when the data is examined in relation to a larger population. Conversely, a small incidence rate within the region may actually be a bigger problem than initially thought when placed in comparison with other regions.

Student teams should note that the data presented in this problem are *discrete* data that capture an entire year of pregnancies and births for the region, during the years specified. The data summarize 365 days of information, of which only one small snapshot is provided. We have no way of knowing when each of the pregnancies or births occurred, simply that they occurred. Many student teams correctly recognized that trend information would provide intra-regional insights. They then proceeded to charge forward by fitting a *continuous* curve $p(t)$ to the pregnancy data using regression or polynomial curve-fitting techniques, presumably to provide a function that they could use to interpolate between the data points. Curiously however, they used $p(t)$ to calculate and evaluate the *continuous* derivative $dp(t)/dt$ at times $t = 0$, 1, and 2 years to get at rate of change information. Setting aside the issue of whether or not such an instantaneous derivative actually exists at these points, this approach tacitly assumes that all points of the surface $f(t)$ in the domain interval $t \in [0, 2]$ have a valid interpretation. This is a mathematical issue similar to the concern regarding significant figures: the data does not have this degree of resolution required to make such an interpretation for rate of change information. A better way of identifying this trend information would be to simply calculate the discrete ratio $\Delta p/\Delta t = (p_2 - p_1)/(t_2 - t_1)$, between the points (t_1, p_1) and (t_2, p_2) for example, to investigate the rate of change between time periods. This ratio is also know as a *first divided difference*.

Thankfully, teams did not invest in active discussion concerning curve-fitting metrics such as R^2 with such a small data set, particularly if they used a quadratic polynomial to approximate the data, since an $(n-1)$th degree polynomial curve can be made to go through n data points *exactly*.

Another general observation worthy of note was a shortage of effective graphics. Teams generated many interesting comparative statistics concerning the adolescent pregnancy issue, yet they chose to present their results in text format. The differences and similarities they wished readers to understand often required a healthy amount of reading and reflection using text, whereas the same ideas could be quickly and effectively understood using a graphic. An effective picture *is* worth a thousand words. However, one word of caution: do *not* simply dump a graphic into your technical report without telling the reader what *you* think it means. Do *not* leave it up to a reader to interpret your graphic.

Finally, many teams relied on credible Internet resources such as www.census.gov. More importantly from an educational prospective, these teams properly documented these sources. This is a very positive trend in papers that are making it to the National level. Similar comments are being heard from Regional judging sessions as well. Student teams are getting increasingly selective in websites they trust for credible information, and they apparently are off to a great start to recognizing intellectual property while giving proper recognition to other people's work, a quality that will serve them well in college and later life. Keep up the good work!

HiMCM Judges Commentary: Problem B: The Skyscraper Problem

Frank Giordano

Naval Postgraduate School

The judges were impressed with both the quality of the analysis and the writing by the teams modeling the Skyscraper Problem, as evidenced by the designation of 9 teams as National Outstanding. The judges commented that the statement of assumptions with justification, style of presentation, and depth of analysis were the best they have seen so far in the four contests.

One of the items that discriminated the better papers was the satisfaction of the requirement to "Solve your model for X = 15, 30, and 90 minutes." Another was the analyzing of skyscrapers both as they are currently built and equipped, and the way the teams felt structures should be built and/or equipped in the future. Data was needed to operate most models developed. Teams varied in the reliability of the data they incorporated. While some teams used educated guesses, others used building codes, results of documented experiments, and so forth. Verification of models was also an important discriminator. Some tested their models to see if they made "common sense." Others compared their predictions with historical results they were able to obtain. One of the major discriminators in the very best papers was the depth with which teams analyzed the evacuation speed of the various devices. Some merely estimated average velocities while others considered traffic densities, overflow situations, turbulence due to mix of various travel speeds, and so forth.

Some things the judges listed as things they would like to see in future contests include: Annotation of computer programs if included, careful definition of inputs required and outputs generated of computer programs, an increase in the use of graphical displays with interpretations, demonstrations with very simple problems before employing the same logic with a detailed computer simulation, and increased documentation of sources used—careful annotation of material used from references with an explanation of how equations that are incorporated from various references follow the assumptions that the modelers are making. And perhaps most importantly, judges desire a careful explanation of the model design, getting from the assumptions to the model.

The judges commend the teams for a truly outstanding job on a difficult problem.

Problem B Summary: Arkansas School for Math and Sciences, Hot Springs AR

Advisor: Bruce Turkal

Team Members: Jamila Amarshi, Michael Herring, Andrew Spann, Daniel Young

We modeled the evacuation of a skyscraper as a dynamic network flow problem. This approach attempted to find a bottleneck that would limit the flow of people escaping from the building. Once we used the model to determine the nature of the flow of people, we attempted to account for the added effects of the fire department aiding with the evacuation and rescue. The end result of our efforts was a computer program capable of simulating the success or failure of the evacuation given an adjustable set of variables that included the number of floors, average number of people per floor, number of stairwells, time limit, and rescue options available to firefighters. We derived an algebraic equation from our computer program and, after fixing simulation values for the necessary variables, used the equation to find the maximum number of stories a building can have and still be evacuated in under 15, 30, or 60 minutes. We found that the maximum height of a building that can be completely evacuated in 15 minutes is one with 22 floors (approximately 88 meters tall). For 30 minutes, the maximum is 45 floors (approximately 180 meters), and for 60 minutes the maximum is 92 floors (approximately 368 meters). We assumed for these maximum height calculations that the buildings had 250 people per floor and 4 stairwells.

Problem B Summary: Illinois Mathematics and Science Academy, Aurora IL

Advisor: Ronald Vavrinek

Team Members: Daniel Gulotta, Bradley Kay, Jered Wierzbicki, Kevin Yang

Initially, we set out to model the interior of a skyscraper, especially its floor plans and stairwells. After researching several buildings, we created a model with a square horizontal cross-section 49.7 m on each side and four stairwells. From this side length and the fact that the four stairwells serviced all floors, we estimated that the occupants take at most 75 seconds to reach a stairwell from any point in the skyscraper.

In our final model, each stairwell extends from the top floor to the ground floor and has a landing at each floor, a landing between each pair of floors, and 12 steps between landings. Through our research we determined that three people could fit comfortably across the width of a stair or landing, so we represent traffic in three "lanes" going down the stairs. The occupants of each floor line up outside the appropriate stairwell door and wait. Whenever a space adjacent to the door opens, the next person in the queue enters the stairwell in the outermost lane.

Once in the stairwell, a few constraints govern peoples' motion. First, everyone moves down the staircase at a steady rate. However, to reflect differences in speed, a random number generator assigns each person a reasonable speed (no more than a jog) for descending the stairwell when unobstructed. When obstructed, a faster person can move into another lane if there is room. Also, people tend to move to the farthest inside lane, away from the walls and doors, so as to take a shorter path and make room for slower people to enter the staircase from lower floors. Finally, there is a low chance that a person will stumble, stopping movement in their lane for a number of iterations while they reorient themselves.

Given the height of the skyscraper in stories and minimum and maximum occupancies for each floor, the simulation determines how long it takes to completely evacuate the building when everyone is lined up at stairwell doors. Adding to this the longest possible time to reach a stairwell gives an excellent approximation of the time to evacuate the building fully. After running several trials with different heights and occupancies, we determined a relationship (with $r^2 = 0.9976$) between the height of the building in stories (H), the number of occupants (P), and the total time for evacuation in minutes (T):
$T = 0.075273*H + 0.001638*P + 1.25$.

Using this formula and the average value of 300 workers per floor that we determined from our research, we found that in 15 minutes, a building of 27 stories or less may be evacuated. In 30 minutes, a building of 56 stories or less may be evacuated. Finally, in 60 minutes, a building of 114 stories or less may be evacuated.

Problem B Summary: Maggie L. Walker Governor's School, Richmond VA

Advisor: John Barnes

Team Members: Andrew Carroll, Victoria Chiou, Jessamyn Liu, James Ware

Our model assumes a uniform population in the skyscraper; everyone travels at the same rate, and the rate at which people move down the stairs is a function of how crowded the stairs, are. Our model divides the evacuation process into three main components: (1) travel on a horizontal plane towards an internal exit, (2) travel through the exit onto the stairs, (3) descent on the staircase and travel on a horizontal plane to an external area of refuge.

We identified variables necessary for the determination of movement rates for each component of the process. A system of difference equations to relate those variables to the number of people evacuated as a function of time was created. Values found in various publications were substituted for variables in the model, in order to create a model that generated actual results. The number of people on a floor, the number on a flight of stairs, the density of people on the flight of stairs, and the speed of people on the stairs was calculated. We calculated this for each floor, for each second that the model covers. Since the model has to go for 60 minutes, we used Excel to run all of the calculations and to produce graphs.

It was determined that the highest a building can be and be evacuated in 15 minutes is 4 stories. Such a building does not qualify as a skyscraper, six stories of height. In 30 minutes, a 7-story building can be evacuated. In 60 minutes, a 14-story building can be evacuated. It was determined that in order to increase the height of a building, one must increase the number of staircases that are available for evacuation. A greater number of staircases results in a lighter distribution of people on the staircases, allowing quicker evacuation rates.

Problem B Summary: Maggie L. Walker Governor's School, Richmond VA

Advisor: John Barnes

Team Members: Benjamin Easter, Konstantin Lantsman, Eric Nielsen, Devin Yagel

Our model, despite its unusual shape (somewhat like two ziggurats, one placed upside down on top of the other) ameliorates several flaws in current skyscraper design. Skyscrapers today often have poor evacuation procedures. In addition, if the stairs through one level of the skyscraper are destroyed, the tenants of the skyscraper who inhabit offices above the problematic floor are stranded. In the event of a fire, bomb, or building collapse, these persons have had no escape. Our model includes simple evacuation procedures using four internal sets of staircases and four external sets. Every inhabitant can quickly and easily access a stairwell that allows them to follow the evacuation route. The external stairs allow persons above the disaster to escape quickly.

In order to test the model, worst-case scenarios were considered. Using population density data from the Sears Tower, we determined the maximum occupancy of our building. We then calculated the numbers of staircases needed to evacuate every person from the building. For each building scenario, we considered the time to evacuate the building assuming the longest, and hence the most time-consuming path. The number of floors was then limited in order to keep the most time-consuming path under 15, 30, and 60-minute evacuations respectively. When the area for every floor was considered along with population density, the total building population was determined. Even when strict evacuation constraints were put into place, and enough stairs utilized to evacuate everyone from the building within those time limits, less than 3% of the building's area per floor was consumed by staircases. Thus, the staircases moved people rapidly and economically, while utilizing a very small amount of space within the skyscraper.

We also considered a test case using our model in place of the World Trade Center Towers. In the case of the South Tower, total evacuation was only three minutes from completion when the tower collapsed. In the case of the North Tower, evacuation was completed 40 minutes before the Tower collapsed.

Therefore, our model accounts for the basic flaws of current skyscraper design. While many skyscrapers ignore security, evacuation brevity and safety are the top priorities in our design. Safety concerns will be of paramount importance in new skyscraper designs. Our model effectively addresses this most vital concern.

Problem B Summary: Westminster Schools, Atlanta GA

Advisor: Charlotte McGreaham

Team Members: Alok Deshpande, Jeffrey Huong, Imran Saleh, Anthony Waller

To solve this problem, we employed our own innovative methods of measuring the capacity and rate of a stairwell. Borrowing from computers, we named this term "bandwidth" and defined it as the number of people that can pass through a certain point per second. While the rate values of people allow us to determine speeds at clear parts of segments of the tower, bandwidth allows us to simulate the congestion of people at various points in the evacuation route. Our two main points of bandwidth measurement were the threshold between the floor and the stairwell, and the actual stairwell itself. Both the doorway and the stairs have dimensional constraints, and this enabled us to pin down and approximate bandwidth figure. Seemingly easy in conception, we found that bandwidth actually quite complex; but after numerous attempts we came up with our final model. In it, we simulate a stream of people and using our bandwidth formula, we calculate the various times to exit buildings of a given heights. An influx of people causes bottlenecks at various floors, but these are mostly balanced out by the reduction of flow once floors are completely cleared of people. Using bandwidth, we came up with a number for the total population of a building that can be evacuated. Then, using assumptions based on empirical data, we came up with a formula that has population as the input and building height as the output. As for alternate evacuation methods, we created a system, based on our assumptions of elevator logistics, to deploy rescue personnel on elevator shafts to save injured and disabled persons. Overall, we believe our model is accurate because it incorporates proven data with experimental analysis, takes into account the reality of evacuation mentality (anxiety), and is able to describe congestion and bottlenecks effectively.

Problem B Summary: Westminster Schools, Atlanta GA

Advisor: Charlotte McGreaham

Team Members: Koon-Ho Cho, Jana Dopson, Michael Miller, Conor Tochilin

After making a number of assumptions about the behavior of the occupants and the skyscrapers themselves, we set out to determine an accurate model for the flow out of the building.

We considered three different evacuation procedures. We determined that the most efficient procedure (i.e., the one that minimized total evacuation time) was in fact the least structured of the three methods. Our procedure of choice, which we dubbed the "Go Method," describes a realistic scenario of evacuees hurrying onto stairways in a fairly disorderly manner. It always yields a total evacuation time less than other, more regulated procedures.

Using the knowledge we gained from evaluating simpler scenarios, we determined a piecewise function to represent the flow of people from the skyscraper. We determined each part of the function and connected them to figure out the total amount of time to evacuate a given building.

This enabled us to define a function representing total evacuation time in terms of all unknowns. The composition of this function could be approximated in terms of several slightly simpler functions. After one round of simplifying, we were left with:

$$T = \frac{1}{F_B(T_0)W_B}\left[P - \frac{T_0}{2}F_B(T_0)W_B\right] + T_0 .$$

Then, relying on some basic properties of flow, speed, and density, along with an empirically determined density and speed relationship, we found the maximum flow in terms of constants. Using data from a number of others' trial evacuations, we were able to formulate a relationship that describes the total evacuation time in terms of building height, population density within the building, and width of the stairs that the people used to egress:

$$T = \frac{gh}{1.005W_B} + 20.5 .$$

Finally, we used data from several theoretical situations to figure out approximate dimensions of an example building that could be evacuated in certain different periods of time.

Time	Height	People/floor	Cumulative stair width
14.437852	17	100	2
29.985116	50	143	4
59.569338	100	250	7

The model could easily be tested, and most of the simplifying assumptions we used end up to have a negligible possible impact on the accuracy. While our estimates, of course, tend to be slightly optimistic (i.e. given time, building height will be overestimated) for any situation, our model generally produces useful and informative results.

Problem B Summary: The Charter School of Wilmington, Wilmington DE

Advisor: L. Charles Biehl

Team Members: Jason Chu, Brian Duncan, Sheel Ganatra, Matthew Williams

We initially state and research some assumptions that make our problem realistic and quantifiable. We then design a mathematical model to evacuate buildings based on a queuing premise; every stairwell is treated as a concatenation of staircase queues and floors. Testing this model with over 180 different combinations of height, side-length, occupancy rate, and number of stairwells gives us the maximum possible building dimensions, occupancy rate, and population for escape times less than or equal to $X = 15$, 30, and 60 minutes. Using this model, the queue-chain model, we find that for $X = 15$ minutes, 4,730 people can occupy a building that is 95 stories with an 85% occupancy rate; for $X = 30$ minutes, 7,770 people can occupy a building that is 105 stories with a 95% occupancy rate; and for $X = 60$, 17,024 people can occupy a building that is 112 stories with a 95% occupancy rate. Finally, we make conclusions as to the accuracy of our model, and propose further recommendations for study of this problem.

Problem A Paper: Montpelier High School, Montpelier VT

Advisor: Sue Abrams

Team Members: Mary Campo, Aaron Hartmann, Lindsay Herbert, Brian Whalen

PROBLEM RESTATEMENT:

The director of a health and environmental control department is concerned with the region's teen pregnancies. Our objective is to determine the possibility of a teen pregnancy problem by analyzing the given data, our independent research, and our mathematical model. In addition, we incorporated our findings into a news article that appeals to adolescents.

ASSUMPTIONS & VARIABLES

- 18 & 19 year olds are the only ones who have graduated or have had the opportunity to graduate.
- A married mother marries in the same social class as herself. This is a strong generalization yet there is no way to include the father's income as an independent variable because the income spread is so great among potential fathers.
- A dropout's average full-time income is half that of a high school graduate.
- 21% of females drop out of school because of pregnancy (79% continue).
- Our risk factor is based on income generating possibilities of the household because you can't make assumptions about the quality of parenting (one vs. two parents).
- The data and statistics are analyzed as of now, not long-term possibilities (continuing high school after dropping out, going on to college, etc.).
- The problem is defined as a child being born into a life in which it cannot be supported.
- The ratio of married to unmarried teenage mothers remains the same over the span of the three years based on their age group.
- The criteria for different risk levels are given below.

HYPOTHESIS:

By analyzing the collected data, we decided that a problem is present. However it has been decreasing since 1998.

MODEL:

To establish if there is a problem we first had to establish which teens are "at-risk." We decided that there are different levels of risk. While a single 12 year old with a baby and a married 18 year old could both be seen as "at-risk," there is obviously a large risk differential. We established three levels of risk: high, medium, and low. To decide which category people fit into we established scenarios. Our scenarios had to be based on variables that are known or easily proven based on statistical analysis. The variable we found with which we could establish a division of people into risk levels is money. We decided to assume that a husband has a similar background and economic status as his wife, and their income is reflected as so. Thus if the wife is a high school dropout, the husband is as well. While we realize that a woman could marry a deadbeat or could marry a millionaire, in general married couples are of similar background and economic status, and we had no way to incorporate wealth of the husband as a distinct variable. Based on findings, we assumed that a high-school graduate's earnings are double those of a high-school dropout, and so a single high-school graduate with a child is economically equal to a married couple of high-school dropouts with a child, and according to our model is at equal risk. Because we were not given marriage numbers for 1998 and 1999, we used the rates that we could establish from 2000. For example, 2834 of 5382 18–19 year old mothers are unmarried. 2834/5382 = 0.5625, so 56.25% of 18–19 year old mothers are unmarried.

We established the high-school dropout rate of teenage mothers as 21%, giving us three variables to use when weighing risk: marital status, high-school graduation status, earning based on education received. Our risk scenarios are:

- High Risk: Not married/Not a high-school graduate;
- Medium Risk: Married/Not a high-school graduate or Not married/High-school graduate;
- Low Risk: Married/High-school graduate.

Our scenarios are based strictly on family income. To every dollar a high-risk family makes, a medium-risk family makes two, and to every dollar a medium-risk family makes, a low-risk family makes two. Using our risk criteria we isolated teenage mothers into groups and correlated this data to establish if there is a problem of teenage pregnancy.

There are many beneficial aspects of our model. It is important that we incorporated varying levels of risk because oversimplification can be dangerous. People of different age groups, marital status, and education obviously are at different risks, and to lump them all together could skew our findings. Lumped together, the trend could show a decline in teenage pregnancies, while broken down, we could find that high-risk mothers are rising while low-risk mothers are declining. This much more exact information would be beneficial to social workers and would allow them to focus on the risk levels and age levels that need the most help. Our model is well defined and easy to understand, thus lending an even greater hand to social workers. The fact that we have broken risk levels down by age allows for even greater pinpointing of a problem.

Our use of national statistics is an accurate way of establishing graduation rates as well as income and allows our model to be used on data from anywhere in the country.

Weaknesses in our model lie in the fact that we were forced to make assumptions. Our graduation rates are not 100% accurate because they are national and don't focus on this region, but this was impossible to do because we do not know the source of the data. While our assumption that a husband is of equal background and economic class may be an overgeneralization, it was necessary because no other assumption is fair. We also failed to incorporate divorce rates into our findings but believe this is not a problem because our findings look solely at the present condition of the mother and in no way are we making a forecast of the future. This is apparent in the fact that we categorized people 17 years and younger as non high-school graduates even though they may still be in school. Our findings are just snapshots of the year in which the data was taken. Correlation and predictions can be made from trends in our findings and not from our results on their own.

ANALYSIS:

To determine if there is a problem, we must first figure out what problem we are looking for. The main problem is that the child isn't being born into a safe environment, not being able to be supported by the parents. Teen pregnancies that generate a detrimental environment for the child are considered problematic. Boundaries need to be set on what is risky. Using our data and our risk assessment boundaries we established which mothers/families are high, medium, and low-risk.

When separating mothers/families into risk levels we used our criteria in conjunction with our facts based on these criteria. Looking at a given year we established how many from each age group would go into a certain risk group. For example:

18–19 year old mothers in 2000:
Births: 5382 Births(unmarried): 2834

Low Risk: Married/High-School grad
5382 – 2834 = 2548 married mothers/families
2548 x 0.79 = 2013 married and graduated mothers/families

Medium Risk: Married/Non High-School grad
5382 – 2834 = 2548 married mothers/families
2548 x 0.21 = 535 married and non-graduate mothers/families

Not Married/High-School grad
2834 unmarried mothers
2834 x 0.79 = 2239 unmarried and graduated mothers/families

535 + 2239 = 2774 Medium-Risk 18–19 year old mothers/families

High risk: Not Married/Not Graduated
2834 unmarried mothers
2834 x 0.21 = 595 unmarried not graduated 18–19 year old mothers.

By this process we established the number of high, medium, and low-risk mothers/families for each age group for each of the three years of data we were given (Tables 1–3).

	Age Groupings	High Risk	Medium Risk	Low Risk
Number of Teens Who Qualify for Certain Degree of Risk	10–14	210	121	0
	15–17	2191	1031	0
	18–19	596	2732	1836
Totals:	All ages	2997	3784	1836

Table 1: 1998

	Age Groupings	High Risk	Medium Risk	Low Risk
Number of Teens Who Qualify for Certain Degree of Risk	10–14	189	19	0
	15–17	2073	975	0
	18–19	623	2851	1917
Totals:	All ages	2885	3845	1917

Table 2: 1999

	Age Groupings	High Risk	Medium Risk	Low Risk
Number of Teens Who Qualify for Certain Degree of Risk	10–14	188	18	0
	15–17	2058	975	0
	18–19	595	2774	2013
Totals:	All ages	2841	3767	2013

Table 3: 2000

We can analyze our data on many levels: by age group, by risk level, and overall. Looking strictly at an age group such as 15–17 year olds we see that both high- and medium-risk went down slightly while low-risk has remained at 0. This is a good sign for 15–17 year olds yet also illustrates that there has been very little change: high-risk has decreased by 133 mothers, and medium-risk has decreased by 56 mothers/families over the past three years. While this is improvement, it is subtle and illustrates that a problem is present.

Analysis based on risk level shows little change over the past three years (Table 4). High-risk mothers were at 35% in 1998 and dropped to 33% in 2000. This illustrates the strength of our model because on the surface we see little change, yet our data is broken down, and problems can be pinpointed.

Percentage of Teens Who Qualify for a Certain Degree of Risk

	High Risk	Medium Risk	Low Risk
1998	35%	44%	21%
1999	33%	45%	22%
2000	33%	44%	23%

Table 4.

Overall our data show that in most areas the numbers of teenage births are somewhat constant, be they high-, medium-, or low-risk. We can pinpoint certain areas such as the number of high-risk 15–17 year old mothers/families as an area of improvement, while we also shine a light on the fact that the number of low-risk 18–19 year old families has increased. This ability to pinpoint problem areas will be extremely beneficial to people using our findings in order to determine where the most support and change is needed. Our findings also allow people to zero in on high-risk groups and allocate support accordingly. Our model is beneficial in that it takes into account degrees of the problem and important information to those working to fix it.

Control #179

TEEN TALK

Volume 1, Issue 1 November 2001

Kids Raising Kids

When is young too young?

By: Team 5

The issue of teen pregnancy had been raised in our area. Is it a problem? We think so. One question you must ask yourself, is when is young too young when deal with the raising of children? A child should not be born into an unhealthy environment. The next question is, what is unhealthy? Certain criterion is used when grouping teen pregnancies into different risk levels. Three risk levels can be formed to group pregnant teens by analyzing their income possibilities and educational background.

High Risk	Medium Risk	Low Risk
Not Married/ Not Graduated	Married/Not Graduated OR Not Married/Graduated	Married/Graduated

FACTS AND FIGURES...

1) 13% of all US births are to teens[1]

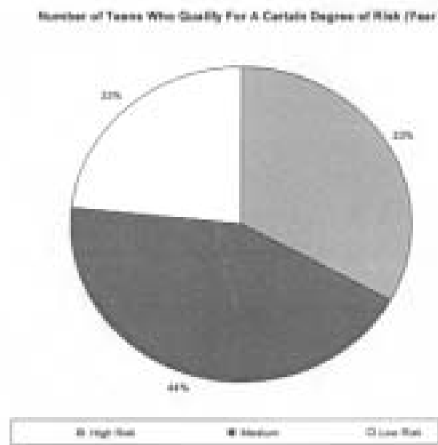
2) ¼ of teenage mothers have a second child within two years of their first child[2]

3) 21% of teenage mothers drop out of school[3]

4) 8% of teenage fathers drop out of school[4]

5) A dropout's avg. income is only half that of a high school graduate[5]

A risk is based purely on how much money they make. A low risk family makes four dollars to every two that a medium risk parent/family makes. Similarly, a medium risk parent/family can make twice as much as a high-risk parent.



In the graph above, all teenage pregnancies from our area have been categorized into the three risk levels. The majority, 44%, of teen pregnancies is at medium risk with 3,767 different cases. Following close behind with 33% and 2,841 different cases, are the high-risk pregnancies. The minority, 23%, of teen pregnancies in our area is classified as low risk. Although lagging by a mere 11% high-risk pregnancies are still a larger problem in our community than medium risk. A threat is posed to each high-risk parent because, without a spouse or high school diploma his or her earning possibilities are severely crippled. Without a sufficient income, these high-risk parents cannot supply a healthy environment for their children. A problem has been defined, and it must be addressed. The responsibility to prevent such teen pregnancy problems lies in the hands of the teens themselves. With so many ways to prevent unwanted pregnancies, sexual education is imperative. Public organizations, such as Planned Parenthood, can supply free aid to any teens that are sexually active but do not wish to become parents. For the health center nearest you, call toll free 1-800-230-7526.

[1] http://www.angelfire.com/ri/apriyouth/index6.html
[2] http://www.angelfire.com/ri/apriyouth/index6.html
[3] http://www.usdoj.gov/kidspage/getinvolved/12_1txt.htm
[4] http://www.usdoj.gov/kidspage/getinvolved/12_1txt.htm
[5] http://www.usdoj.gov/kidspage/getinvolved/12_1txt.htm

Problem B Paper: The Ellis School, Pittsburgh PA

Advisor: Eric Zahler

Team Members: Mary Thibadeau, Anita Vin, Cynthia Wu

PROBLEM RESTATEMENT:

The goal is to determine how tall a skyscraper can be for a given evacuation time. By combining our assumption of maximum number of people per floor with calculated height, we find maximum occupation.

INITIAL PLAN:

Time (X) is the independent variable.

Height (h) and Occupation (p) are dependent on time.

Floor size determines number of exits, which determines building height, which determines occupation.

We searched the Internet for government skyscraper regulations, but found nothing useful.

We made certain simplifying assumptions. To calculate speeds, we timed ourselves walking (either across a hall or down steps) and found an average.

ASSUMPTIONS:

1. Everyone tries to exit as soon as possible (i.e., no one turns back).
2. Everyone knows the closest exit and follows an escape plan.
3. Handicapped people are carried by others. There are relatively few handicapped people, so this does not slow traffic.
4. The distance from the floor of one floor to the floor of the next is 4.59877 meters (see below for justification).
5. Everyone walks at a speed of 1.5 m/s to get to the stairway or out the door. This was found by timing a person walking.
6. The skyscraper is a rectangular prism with a square base, which has an area of 3391 m^2 (based on the USX Tower in Pittsburgh).
7. Each floor is identical except for the ground (first) floor, which is a lobby. Few people work in the lobby. They exit before others get down the stairs.
8. There are no underground floors.
9. The number of people working on a floor is the maximum occupancy of that floor.
10. Doors leading to the outside are at the end of the staircases.
11. When evacuation begins, every worker is at the door of his/her cubicle.
12. People from a floor walk down stairs in pairs, except for the last person.
13. To accommodate a stairwell long enough that the stairs are not too steep, the stairwells (which lie mainly outside the building) protrude into the building a distance equal to the width of a cubicle.
14. People on the top floors are in the most danger. If they don't go down quickly, escape routes may collapse.
15. Firefighters enter the building only to put out fires or to rescue people. Their only mode of travel is the elevator.
16. Figure 1 is a typical floor. There are no bathrooms, lounges, or other landmarks. The square workspaces are filled with 2.21 m x 2.21 m cubicles, each of which holds one person. (See below for explanation of cubicle size.)

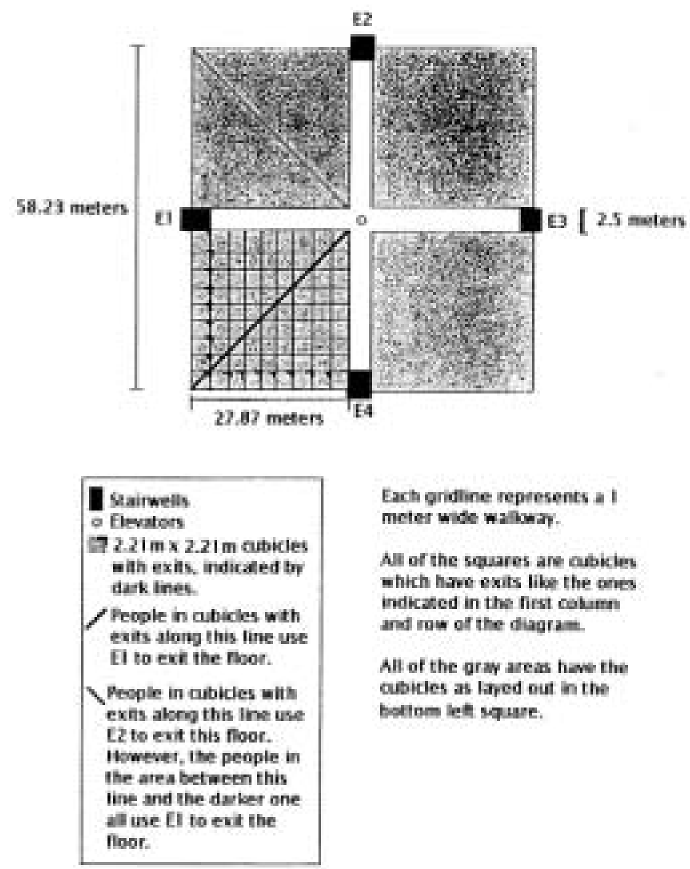


Figure 1

17. Two main hallways in a cross shape (see below for explanation) lead to four separate stairwells on each floor. Each main hallway is 2.5 m wide.
18. Each cubicle is surrounded by 1 m wide walkways unless bordered by a main hallway or a wall.

REASONING AND MODEL:

To find walking speed down stairs, we used a sample staircase similar to one in a skyscraper. Its vertical distance is 325.12 cm; the average time is 6.27 s. Dividing 3.25 m by 6.27 s gave an average vertical speed of 0.518 m/s. Horizontal speed was not considered because net horizontal distance is 0 since the stairs turn 360° between floors.

Next, we found floor-to-floor distance. We found a table of well-known skyscrapers that gave heights and numbers of floors. We divided each height by the number of stories. The average of these lengths is 4.599 m. This accounts for floor thickness; thus, multiplying by the number of floors gave the height of our building.

The next step was to determine a floor plan that incorporated hallways, work areas (comprised of cubicles), stairwells, and elevators. Here our goals were that workers get to a stairwell in reasonable time and that no stairwell accommodates more people than others do. Moreover, although it is not our focus, the elevator should be in a central area to accommodate daily traffic. We decided on a plan with four stairwells. This ensures equal traffic flow down each stairwell. Because of the cubicles, we decided that all hallways must travel either North-South or East-West. In other words, no one can cut diagonally across an area to access an exit. We wanted a layout that minimized maximum travel distance to a stairwell, so we decided against stairwells in corners. With corner stairwells, a person in the center walks about 55.74 meters. Corner stairwells might also suggest a diagonal hallway design, which does not work with a cubicle layout. We also decided against interior stairwells. Although this might reduce the greatest distance traveled, it increases the possibility of congestion as everyone must funnel into first-floor hallways to get out of the building.

We decided on two major hallways down the middle of the floor (creating a plus sign), leading to four stairwells, one at the middle of each side. This gives a maximum travel distance of 27.87 meters, which is further reduced if stairwells are recessed into the building. This plan divides the building into four equally sized, square work areas. If one divides the building into four triangles, each of the stairwells accommodates the cubicles with openings in one of the four triangles (see Fig. 1). We decided to surround each cubicle with smaller hallways to provide access to major hallways and stairwells. Each cubicle has an opening in one corner only, which helps to eliminate confusion as people exit and reduces the furthest distance traveled.

To find maximum occupancy, we needed the number of cubicles on each floor. We found the areas of the major hallways by multiplying their width (2.5 m) by building length (58.23 m). After multiplying by 2 to account for two hallways, we subtracted the 2.5 m x 2.5 m overlap to get hallway area of 284.9 m^2. We subtracted this from the floor area (3391 m^2) to get office space of 3106.1 m^2. Since a hallway divides a floor into four equal office spaces, we divided by 4 to get 776.525 m^2. The square root of this (27.87 m) gave the length of the small square area. From this we found the number of cubicles.

The cubicles along a small square area's side are separated by hallways 1 m wide. The initial size of the cubicle we used to estimate the maximum occupancy per floor (3.925 m^2) was obtained from the Internet. Combining these two pieces of data, we split the length of the small square area into subunits of 1 cubicle + 1 hallway unit, each with a length of 2.98 m. This division left one corner cubicle without a hallway counterpart, so we subtracted the length of a cubicle's side from the length of a side of the small square area to get 24.89 m. We divided this length by the length of a cubicle + hallway subunit to get 8.34 cubicles, which we rounded to 8. We then added the disregarded cubicle to obtain 9 cubicles along a side. We squared this number to get 81 cubicles per small square area. Since there are 4 small square areas, there is room for a maximum of 324 cubicles. Thus, maximum occupancy per floor is 324 people.

Since we rounded the number of the cubicles down, we could not use the cubicle size from the source (3.925 m^2). Instead, we subtracted 8 m (the width of the hallways) from the side of the small square area (27.87 m). We then divided the result (19.87 m) by 9 (the number of cubicles along one side) to get a cubicle size of 4.874 m^2, or a square cubicle with 2.21 m sides.

At first, we assumed there are no people in front of the person who is farthest from the stairways from the highest floor. We developed the equation

$$X = \frac{25.656\text{m}}{1.5\text{m}/\text{s}^2} + \frac{(F-1)4.598774\text{m}}{0.5183\text{m}/\text{s}},$$

where F is the number of floors and X is the time for the last person to exit the building. The 25.656 m is the greatest possible distance from a cubicle doorway to its exit, which is less than 27.87 m (one side of the small square work area) because the stairwells protrude into the building by a cubicle (2.21 m). So we subtracted 2.21 m from 27.87 m to get 25.66 m. We divided by 1.5 m/s^2 to get the time for a person to travel from the furthest cubicle to the exit. In the second part of the equation, we found the time required to go down the stairs. We subtracted 1 from the top floor number since one has to travel down F–1 flights of stairs. We then multiplied by the vertical distance per flight to get total vertical distance traveled. We divided this by 0.5183 m/s (vertical speed). Then we added the two expressions to get the time for the last person to exit (X). Using 15 minutes for X, we found F to be about 100 floors. Unfortunately, this model ignores congestion.

Our next step was to lessen congestion. As an example, we calculated time for a person closest to the third floor stairwell to get to the second floor. From this we calculated time for the person farthest from the second floor stairs to get to the stairs and found that this overlapped the time for people on the third floor to get to second floor. To lessen congestion, we tried requiring that each floor use only two stairways. Unfortunately, the additional walking and waiting times for people farthest from stairways creates congestion. The person from the higher floor who is closest to the stairway reaches the next level before the previous level evacuates their floor; the large influx of people walking towards an exit causes congestion at the exit doorway on each floor. So we decided to have all floors use all stairways.

We then looked at congestion at stairway doors. To find wait time at a stairway, we assumed that no one from another floor is in the way. First we calculated the number of people who travel the same distance to get to a stairwell. For example, there are 9 people who travel a distance of 8 cubicles and across 8 hallways and cause congestion by getting to the door at the same time. We got these results from the floor plan (see Fig. 1). Those in cubicles with exits on the largest diagonal travel 8 cubicle-hallway units, those on the next largest diagonal travel 7 units, and so on. Then we took into account the triangular counterpart on the other side of the main stairway (see Table 1, Columns A and B).

A	B	C	D	E	F	G	H	I
0	2	0	0	0	0	0	0	8.87
1	4	3.21	2.14	0.77	2.91	–2.14	2.91	11.8
2	6	6.41	4.28	1.54	5.82	–1.37	5.82	14.7
3	8	9.62	6.41	2.32	8.73	–0.59	8.73	17.6
4	10	12.8	8.55	3.09	11.6	0.18	11.8	20.7
5	12	16	10.7	3.86	14.5	0.95	15.7	24.5
6	14	19.2	12.8	4.63	17.5	1.72	20.3	29.2
7	16	22.4	15	5.4	20.4	2.49	25.7	34.6
8	9	25.7	17.1	2.7	19.8	3.26	28.4	37.3

Table 1:

Key:
A: Number of cubicle-hall units traveled (also group letter)
B: Number of people in group
C: Distance between cubicle and door (meters)
D: Time to go from cubicle to door (seconds)
E: Time last pair waits to get on stairs if only this group travels on stairs (seconds)
F: Total time for last person in group to get from cubicle to stairs (seconds)
G: Wait time if no one else is in front except previous group (seconds)
H: Time for last pair in each group to get from cubicle onto stairs (seconds)
I: Time for group to go from cubicle to door of floor below (seconds)

We calculated the distance and time for each group (organized by the distance traveled from their respective cubicles) to get from cubicle to stairwell. We found the distance by multiplying the number of cubicle-hallway units traveled by their length (3.21 m) (see Table 1, Column C). Then, we divided by walking speed (1.5 m/s) to get time (see Table 1, Column D).

We calculated the time the last pair in each group waits to get on the stairs if they do not have to wait for groups before them. We used the expression

$$\frac{[(\text{number of people in group}) - (\text{last 2 people in group})] \times (\text{2 steps between each pair on the stairs})}{(\text{2 people per pair}) \times (0.5183 \text{ m/s})}$$

Essentially, the time the last person in the group waits is the time it takes the previous people to get on the stairs (see Table 1, Column E).

Then we found the sum of the time for the last person in a group to get from cubicle to door and the time the person waits to get on the stairs if there is no congestion at the door. This gave us the time the last person in a group needs to get into the stairwell (see Table 1, Column F).

Next we considered congestion caused by the previous group, A, when the next group, B, arrives at the door. We did this by subtracting the time the first person in B takes to get to the door from the time the last person in A exits the door. If the result is negative, there is enough time before the first person in B arrives for the last person in A to exit. If the result is positive, the first person in A waits that long to enter the stairwell (see Table 1, Column G).

If there is no waiting time caused by group A, then the time for the last person in B to go into the stairwell is the sum of the time to get from cubicle to door and the waiting time at the door for the people in B to get into the stairwell (see Table 1, Column H).

If group B waits for A, there is another concern. The waiting causes a back-up effect for groups behind B. To resolve this, we added the time for last person in B to get from cubicle to door, the waiting time at the door for people in B, and the waiting time for people in A. This is the time for the last person to get from cubicle to stairwell. For later groups, one must find the difference between this value and the time it takes the next group, C, to get to the door. This is the time C has to wait. One can use this value to calculate times for the last people in groups after C to get from cubicle to door. From these calculations, the last person in the last group exits the stairwell door in 28.41 s (see Table 1, Column H). Assuming that everyone keeps moving on the stairs, the time it takes to get from cubicle to doorway of the next floor down is found by adding the time to get down one flight ($4.5988\text{m}/0.5183\text{m/s}^2 = 8.873\text{s}$) to the time from cubicle to when one starts down the stairs. The time it takes the last person to get from cubicle to next floor down is 37.28 s (see Table 1, Column I).

We assumed that top floors are in the most danger and should be evacuated quickly. Thus, we decided that people on lower floors should wait longer. We determined that the first few groups on each floor should go because they don't bump into one another on the stairs. After these groups enter the stairs, everyone except those on the top floor stops and waits until the floors above pass. So the entire top floor goes, and as soon as the last person passes the doorway to the next floor down, the rest of the people on that floor go, and so on. The last person to exit the building is the one farthest from the stairs on the second floor. We calculated the time it takes this person to get out of a building with F floors as follows:

X = time down 1 flight + time from farthest cubicle on top floor to next floor down + time waited for higher levels

$X = 8.873 \text{ s} + 37.28 \text{ s} + (37.28 \text{ s} - 17.104 \text{ s}) * (F - 2)$

$= 20.176\, F + 5.801 \text{ s}$

For X = 15 min, F is 44 floors. Multiplying by the maximum people per floor gives a maximum capacity of 14,502. Multiplying the number of the floors by the floor-to-floor height (4.4988 m) gives a building height of 202 m.

For X = 30 min, F is 88 floors. Maximum capacity is 28,512. Height is 405 m.

For X = 60 min, F is 178 floors. Maximum capacity is 57,672. Height is 818.6 m.

If a person is the last to exit floor B without the complication of people from other floors, congestion begins with the group (call it G) that travels 4 cubicle-hallway units. This person endures a wait on floor B caused by G and all behind them. However, before the last person in G enters the stairwell at 11.816 s, the first person from A arrives at 8.8732 s, stopping G.

After everyone on floor A passes the stairwell entrance on floor B, those from B move into the stairwell. From this point, the time that the last person on B waits to get on the stairs is the same as if no one came down from A. If it is the group after G that is stopped by A, the last person on B waits less. If it is the group before G that stops, the last person on B waits longer. However, because G is the key group in both horizontal and vertical congestion, we can use the difference between the time for the last person to get from cubicle to the next floor and the time to get from cubicle to stairwell. This is the time the person waits for people on his/her own floor and go down one flight of stairs, just as if no one from A interrupted flow.

STRENGTHS AND WEAKNESSES OF OUR MODEL:

Strengths:

Results are reasonable and match real world skyscraper data.

It effectively clears top floors first without excessively compromising the efficiency of clearing lower floors.

Weaknesses:

People on lower floors must wait for higher floors, which may be unrealistic in terms of emergency reaction.

It does not account for slowing factors such as obstacles that may be caused by the disaster event.

It assumes that the top floor is most dangerous, which may not be true of disasters that can occur anywhere in a building.

It has only one elevator for the entire building.

Time constraints did not permit exploring additional stairwells.

SOURCES:

1. Source of our initial average cubicle size: http://www.theofw.com/products.asp?Page=products&load=two
2. List of the world's tallest buildings, including number of stories and height: http://www.infoplease.com/ipa/A0001338.html
3. Source of USX Tower information: http://www.skyscrapers.com

Reprinted from *Mathematics: Modeling Our World* Course 3

Fairness and Apportionment I

Marsha Davis
Pat Taylor

Teacher notes

The activities in this Pull-Out deal with apportionment and the fairness/unfairness associated with an apportionment. In Activity 1 students devise an algorithm for dividing a collection of coins among five people. Activity 2 introduces the Hamilton method of apportionment, once used to apportion the U. S. House of Representatives. Activity 3 focuses on divisor methods of apportionment, which include the Jefferson, Webster, and Adams methods.

Activities in the next MMOW Pull-Out will explore the Hill method of apportionment, the method presently used to apportion the U. S. House of Representatives.

The activities in this Pull-Out are adapted from Fairness & Apportionment, a unit in COMAP's secondary school core curriculum *Mathematics: Modeling Our World* (MMOW), course 3. For more information about MMOW, check COMAP's website: http://www.comap.com.

Examination copies of MMOW courses 1, 2, 3, and 4 are available from W. H. Freeman Publishing, www.whfreeman.com.

Mathematics prerequisites and discussion: In Activities 1–3 students struggle with the problem of apportioning identical indivisible objects fairly, such as gold coins, computers, tickets, and representatives. Students work with percentages, fractions, rational functions, and rounding methods. The functions proposed in Activity 3 are step functions. Graph them and students will see some very unusual graphs not found in standard textbooks.

Materials needed: Graphing calculators.

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Lesson notes

In Activity 1 students discover a method(s) for dividing an estate in which they have to deal with non-integer shares of indivisible objects. Allow time for groups to share their apportionment methods. Pay particular attention to any student's suggestions of ways to measure fairness.

It is important that all students see the round-off method and the method of distributing leftovers according to fractional parts in descending order (Hamilton method). Therefore, if students do not propose these methods, add them to the discussion. Hold off on Exercises 6 and 7 until after the discussion and, if short on time, assign them as homework.

Activity 2 introduces the Hamilton method of apportionment. Make sure that students reflect on the paradoxes exposed by Exercises 2 and 4 and that they can distinguish between them. If you are running short on time, assign Exercise 6 as homework. Exercise 6 will be needed for Activity 3.

Activity 3 introduces divisor methods: methods of apportionment that determine quotas by dividing the population by an ideal ratio (district size) and then adjusting that divisor. Finding an adjusted divisor in Exercise 2 can be a lengthy process. Pat suggests that you let students try a few divisors on their own and then assign several divisors to each group. This way students can discover that more than one adjusted divisor will work.

Pat suggested the use of the TI-83's iPart function and tables in Exercise 4. Make sure students understand that the problem is solved when they find an x value that makes their function y = iPart(P_A/x) + iPart(P_B/x) + iPart(P_C/x) equal to the total number of objects to be distributed, H. You can also have some fun graphing this function. It is a step function, where the steps have unequal widths. If you trace along the function until you see y = H and then zoom in, you will see a horizontal line segment with y-coordinates equal to H. Tracing along this line segment can reveal the interval of adjusted divisors, x, that work.

In Exercise 5, see if students can adapt the function used in Exercise 4 for the Adams method. You may want to give the hint that the Adams method assigns at least one delegate to every state.

After students have worked with all three divisor methods, ask them to look at the Jefferson and Adams methods and decide whether either of these proposed methods appears to favor large or small states. You may want to reveal that Thomas Jefferson, our third president, was from Virginia (large state), and John Quincy Adams, our sixth president, from Massachusetts (small state). Have them compare both methods for Exercises 1 and 6. Time permitting, have students investigate numerical measures of unfairness for the two methods.

Solutions to Activity 1

1 Sample algorithm #1: Find the exact number of coins each heir should receive. Round down if the fractional part of the share < 0.5; round up if the fractional part ≥ 0.5. (This algorithm works for 77 coins, but will fail in a later example.)

Sample algorithm #2: Find the exact number of coins each heir should receive. Assign to each heir the integer part of his/her share. Then assign the remaining coins, in order, to the heirs whose shares have the largest fractional parts.

2 Ideal shares: 7.7, 15.4, 23.1, 11.55, and 19.25, for Anna, Ben, Cindy, Denise, and Ed, respectively. Actual shares for each of the sample algorithms: 8, 15, 23, 12, 9, to Anna, Ben, Cindy, Denise, and Ed, respectively.

3 One criterion of fairness might be that each person receives a share as close as possible to the percent described in the will.

4 Ben will most likely be unhappy. Ben has the largest difference between his ideal share and actual share.

5 Sample algorithm #1: assigns 8, 16, 23, 12, 20, a total of 29 – one too many! So algorithm #1 fails in this case. Sample algorithm #2: 8, 16, 23, 12, 19, a total of 78.

6 a No, Denise's proposal is not fair. The 8 coins should be allotted in proportion to the relative shares of the remaining heirs.

b Sample: The shares of the remaining heirs are in the proportion 10:30:15:25. Since 20 + 30 + 15 + 25 = 90 these numbers are in the proportion 20/90:30/90:15/90:25/90. If 8 coins are apportioned accordingly, the ideal shares are 1.78, 2.67, 1.33, 2.22. Using Sample algorithm #2, we find that the 8 coins should be distributed 2, 3, 1, and 2.

Another approach is to put Anna's 8 coins back into "the pot" and apportion all 77 coins among the four remaining heirs in the proportion 20/90, 30/90, 15/90, and 25/90. This leads to the same final distribution: 17, 26, 13, 21, for Ben, Cindy, Denise, and Ed, respectively.

7 a The population of the school is 675. The ideal share (quota) for each class is 29.6, 25.2, 23.7, and 21.5 tickets, respectively.

b Sample algorithm: Rounding to the nearest integer gives 30, 25, 24, 22, for a total of 101. We take one ticket back from the senior class, giving it 21, because of all the shares rounded up, the seniors had the smallest fractional part. (Answers may vary if other methods are used.)

Solutions to Activity 2

1 a 256 + 372 + 372 = 1000. Ideal ratio = 1000/25 = 40. Ideal shares (quotas) are 25*(256/1000) or 256/40 = 6.4, 372/40 = 9.3, 9.3 for srs., jrs., and soph., respectively. The Hamilton method yields an apportionment of 7, 9, 9 delegates for the srs., jrs., and soph., respectively.

b The ideal ratio of total students to total representatives is 1000/25 = 40 students per representative. The actual ratios are 256/7 = 36.6, 372/9 = 41.3 and 372/9 = 41.3. The jrs. and srs. have less representation (41.3 students per representative versus 36.6 students per representative).

2 a

	25 Delegates			26 Delegates		
	Srs.	Jrs.	Soph.	Srs.	Jrs.	Soph.
Population	256	372	372	256	372	372
Ideal number of students/delegate	40	40	40	38.46	38.46	38.46
Ideal number of delegates (quota)	6.4	9.3	9.3	6.66	9.67	9.67
Number of delegates assigned	7	9	9	6	10	10

b The jrs. and soph. each gained a delegate and the srs. lost a delegate.

c The srs. will be upset because they lost a delegate even though the total number of delegates increased!

3 See 2(c).

4 a In 1990 state quotas were 64.7, 24.7, 10.6. The Hamilton method assigns 65, 25, 10 to states A, B, and C, respectively. In 2000 state quotas were 64.36, 25.25, and 10.40. The Hamilton method assigns 64, 25, 11 seats to A, B, and C.

b State A loses a seat to state C even though A's population has increased and C's has decreased.

5 a P = PA + PB + PC. P/H represents the ideal number of people per representative.

b They represent the actual number of people per representative in each state.

c Sample answer: $\left|\frac{P_A}{a}-\frac{P}{H}\right|, \left|\frac{P_B}{b}-\frac{P}{H}\right|, \left|\frac{P_C}{c}-\frac{P}{H}\right|$.

6 a Distribute 22, 29, and 39 computers to schools A, B, and C, respectively.

b The ideal number of students per computer is 900/90 = 10. The actual number of students per computer is 9.86, 9.93, and 10.13 for schools A, B, and C, respectively. School A got the best deal. It has the lowest number of students per computer. School C got the worst deal. Its number of students per computer is larger than the ideal ratio.

Solutions to Activity 3

1 Smaller. When you divide by a smaller number, you get larger quotients.

2 Sample answer:

Attempt number	Adjusted divisor (x)	P_A/x	P_B/x	P_C/x	Apportionment total
1	9	24.11	32.00	43.89	99
2	9.8	22.14	29.39	40.31	91
3	9.9	21.92	29.09	39.90	89
4	9.85	22.03	29.24	40.10	91
5	9.87	21.99	29.18	40.02	90
6					
…					

3 Any divisor, x, between 9.864 and 9.875 will assign 21, 29, and 40 to schools A, B, and C, respectively.

4 After entering the function, it may be helpful to set TblStart = 10, the ideal ratio, and ΔTbl = 1. Notice that the values you see in the Y1 column are all below 90, so scroll backwards. Using x = 9 gives a value of 99, which is too large. So, make ΔTbl smaller, say ΔTbl = .1. You will discover that the x needs to be somewhere between 9.8 and 9.9. Again adjust ΔTbl and continue until you find an x value that produces a function value of 90. This x will lead to an assignment of 21, 29, 40 to schools A, B, and C, respectively.

5 Use the function
y = 3+iPart(450/x)+iPart(345/x)+iPart(205/x).
The ideal ratio is 1000/25 = 40. Start the table at x = 40 and work from there to find and x that makes the function 25. Note that any x value between 41 and 43.1 will work. (Alternatively, use
y = iPart(450/x)+iPart(345/x)+iPart(205/x)
and determine an x that yields 25 – 3 = 22.) The apportionment is 11, 9, 5 for the soph., jrs., and srs. respectively.

6 Use the function
y = round(284000/x,0)+round(488000/x,0)+round(228000/x,0).
Any x value between 39.15 and 40.57 will work. The apportionment is 14, 25, 11.

Where There's a Will There's a Way

Sometimes it is necessary to divide a collection of things that cannot be cut into pieces, or sold for cash. For example, suppose that three people A, B, and C, want to divide 100 identical gold coins where A's share is 25%, B's share is 30%, and C's share is 45%. No problem: A gets 25 coins, B gets 30, and C gets 45. But if there are 101 coins, nothing comes out even. Who gets the extra coin? How do you decide what is fair?

Fairness and Apportionment

*The allocation of a number of objects among several recipients is called **apportionment**. In cases dealing with apportionment, you should ask the all-important question, "Is this division fair to all the recipients?" In the activities in this Pull-Out, you will develop fair-division algorithms—rules for distributing the objects with the least amount of unfairness.*

1. Suppose there are 77 coins in the Chasten estate. Divide the coins among the five heirs according to the will (see **Figure 1**). Write an algorithm for using your fair division method.

"My entire estate consists of my collection of valuable, identical gold coins that may not be sold or given away.

To my faithful housekeeper, Anna, I bequeath 10% of my estate.

To my children:
my son, Ben, 20%;
my daughter, Cindy, 30%;
my daughter, Denise, 15%; and
my son, Ed 25%.

Figure 1.
The Chasten will.

The will in Figure 1 specifies the exact fraction of the estate each heir should receive. If fractional coins were permitted, the resulting allocation would give all heirs their **ideal shares**—the exact amount or number of items they should each receive as a result of the apportionment. Many times, though, ideal shares involve fractions of objects, which is not allowed. Thus, the amount or number of items the heirs eventually do receive as a result of the apportionment are their **actual shares**.

2. According to the Chasten will, what are the ideal shares of the heirs? According to your division method, what is the actual share each heir receives?

3. What criteria for fairness did you use in deciding on your method?

4. Which of the heirs will most likely feel treated unfairly? Why?

5. Oops, another gold coin has been found. Now there are 78. Use your algorithm to apportion these 78 coins according to the Chasten will. What is each heir's actual share now? How good is your algorithm with 78 coins?

6. One fair-division algorithm says Anna should receive 8 of the 77 coins. Suppose, however, Anna dies before the will is executed.

   a) Denise proposes an easy fix: just distribute the 8 coins evenly among the 4 remaining heirs. Is Denise's proposal fair? Why or why not?

   b) Assuming that Denise's method is not fair, how would you distribute the 8 coins fairly? Support your reasoning numerically.

7. Suppose your school has 200 freshmen, 170 sophomores, 160 juniors, and 145 seniors, and also has 100 free tickets for the championship basketball game.

   a) What is the ideal share of tickets each class should get? Ideal share of an apportionment is usually called **quota.**

   b) What is the actual share of tickets that should be allotted to each class in a fair distribution?

   c) According to your apportionment in (b), which class gets the best deal? Which class gets the worst? What numerical measure did you use to decide how good or bad your apportionment was to each class?

The Hamilton Method

Suppose the student council at a high school consists of a fixed number of delegates apportioned among the seniors, juniors, and sophomores. How would you apportion the delegates among the classes with the least amount of unfairness? (A similar, but more complex, problem is the apportionment of the U.S. House of Representatives.)

One method of apportionment, the **Hamilton method**, is named for the American statesman Alexander Hamilton, who suggested its use. To use this method, first calculate the quotas (ideal shares) and give each class a number of representatives equal to the integer part of its quota. If too few representatives are assigned, arrange the classes in descending order of the fractional parts of their quotas. Assign the remaining delegates in that order until all are assigned.

1. Suppose your school has 256 seniors, 372 juniors, and 372 sophomores, and assigns student council representatives by the Hamilton method. The student council has 25 delegates.

   a) Compute the Hamilton apportionment of the 25 representatives among the three classes.

   Ideal ratio $= \dfrac{\textit{Student population}}{\textit{Number of delegates}}$.

   b) Suppose the classes agreed to measure unfairness in the apportionment as the differences between the actual ratio of students per representative to the ideal ratio for the classes. Discuss the fairness of the Hamilton apportionment.

2. For the situation in Exercise 1, the principal sees there is no absolutely fair apportionment of 25 delegates. He decides to break tradition and enlarge the council to 26 delegates, hoping the addition of an extra delegate will lead to a fairer apportionment.

   a) Use **Figure 2** to record the Hamilton apportionments for 25 delegates, then for 26.

	25 Delegates			26 Delegates		
	Srs.	Jrs.	Soph.	Srs.	Jrs.	Soph.
Population	256	372	372	256	372	372
Ideal number of students/delegate						
Ideal number of delegates (quota)						
Number of delegates assigned						

Figure 2.
APPORTIONMENT OF STUDENT COUNCIL DELEGATES.

   b) Compare the apportionment of delegates in the first case (25) with the apportionment in the second case (26). Describe what happened when the extra delegate was added to the council.

   c) Do you think any of the classes will be upset with the result after adding an additional delegate? Why or why not?

In 1880, the U.S. House of Representatives was using the Hamilton method to apportion seats among the states. The method was based on an ideal ratio, I, namely the ideal congressional district size:

$$I = \text{Ideal district size} = \frac{\text{U.S. population}}{\text{Number of seats in Congress}}.$$

Then for each state,

$$\text{Ideal number of seats} = \text{quota} = \frac{\text{State population}}{\text{Ideal district size, } I}.$$

After the census of 1880, however, a new apportionment was needed. At that time the size of the House was not fixed. As they considered sizes between 275 and 350, a problem affecting the state of Alabama drew much attention. If the house size increased from 299 to 300 seats, Alabama's number of seats would have gone from 8 to 7! How could this have happened when there were more seats available?

FYI:

When something happens in a situation that runs counter to a person's intuition, it is known as a paradox.

This unexpected discovery showed that Hamilton's method has a flaw: a state can lose a seat even though the total number of seats increases! This undesirable possibility is now known as the **Alabama paradox.**

3. How does the student council problems in Exercises 1 and 2 illustrate the Alabama paradox?

You have seen that the Hamilton method, which seems so reasonable, has a serious flaw. There's more bad news: the method has another flaw known as the **population paradox**. See if you can discover it by examining this next situation.

State	1990 population	2000 population
A	6,470,000	6,500,000
B	2,470,000	2,550,000
C	1,060,000	1,050,000

Figure 3.
POPULATION PARADOX.

4. Suppose that 100 seats are to be assigned to three states in 1990 and again in 2000. The populations of the states are shown in **Figure 3**.

   a) Use Hamilton's method to find the apportionment of the 100 seats in 1990 and 2000.

   b) What is paradoxical about the results?

5. Suppose that populations of states A, B, and C are represented by P_A, P_B, and P_C respectively, and the number of representatives to be apportioned is H.

   a) Write an expression for P, the total population in the three states. What does P/H represent?

   b) Suppose that the numbers of delegates assigned to the three states are a, b, and c, respectively. What do the expressions P_A/a, P_B/b, and P_C/c represent?

   c) Write expressions that represent the amount of unfairness in each state's share.

6. A school district has three schools A, B, and C, with students populations of 217, 288, and 395, respectively.

   a) Use the Hamilton method to apportion 90 computers among the three schools. Then complete **Figure 4.**

   b) Which school got the best deal? Which got the worst?

	A	B	C
Population	217	288	395
Ideal number of students/computer			
Ideal number of computers (quota)			
Number of computers assigned			
Actual number of students/computer			

Figure 4.
COMPUTER APPORTIONMENT.

Save your results from Exercise 6 for use in Activity 3.

Divisor Methods

FYI:

*A **divisor method** is a method of apportionment that determines quotas by dividing the populations by an ideal ratio or an adjusted ratio, then applying a specific rounding rule. In such a method, the quotas are adjusted in such a way that the integer parts sum to the total number of items being apportioned.*

Refer to your completed copy of **Figure 4** from Exercise 6, Activity 2. You should have discovered that school C, the largest school, received the worst deal. In looking for another way to divide the computers among the schools, School C suggested using a divisior method.

Recall that the quotas for each school were obtained by dividing each school's population by the ideal student per computer ratio, I, in this case 900/90 = 10 students per computer.

$$\text{Quota for school S} = \frac{\text{School population}}{\text{Students / Computer}} = \frac{P_S}{I}\,.$$

$$\text{Adjusted quota for school S} = \frac{\text{School population}}{\text{Adjusted (Students / Computer)}} = \frac{P_S}{x}\,.$$

The challenge is to find an x so that the sum of the integer parts of the three adjusted quotas totals 90.

1. Using an ideal ratio of 10 as the divisor, the integer parts of the quotas only sum to 88. Do you think that you need to divide by a number larger or smaller than 10 to get the integer parts of the quotient to total the desired 90? Why?

2. Choose a new divisor, then compute the adjusted quotas and their integer parts. Keep adjusting the divisor and re-computing the adjusted quotas until the sum of the integer parts of your adjusted quotas is 90. Keep a record of your trial divisors, adjusted quotas, and apportionment totals in a table similar to **Figure 5**.

Attempt number	Adjusted divisor (x)	P_A/x	P_B/x	P_C/x	Apportionment total
1					
2					
3					
...					

Figure 5.
RECORD OF DIVISORS, ADJUSTED QUOTAS AND TOTALS.

3. What adjusted divisor produced the desired apportionment for your group? Compare the number with the final divisors of other groups in your classroom. Do you all have the same divisor? Why or why not?

NOTE:

On a TI-83, see the MATH, NUM menu for iPart.

4. Here's a more efficient way to solve Exercise 2. Enter the function y = iPart($217/x$) + iPart($288/x$) + iPart($395/x$) into your calculator. Next, create a table and search for an x that makes the function 90. Once you have your x, determine the Jefferson apportionment.

The method suggested by school C is called the **Jefferson method** of apportionment. It is one of many divisor methods, and differs from other divisor methods only in the way the quotients are rounded after the divisors have been adjusted. It rounds by dropping all fractional parts of the quota. Here are two other divisor methods:

FYI:

In 1840, the U.S. House of Representatives changed from the Jefferson to the Webster method of apportionment. The Adams method was never adopted.

The **Adams method**: round the quotients up to the next integer.

The **Webster method**: when the fractional part is 0.5 or greater, round up and when the fractional part is less than 0.5, round down.

5. Suppose that there are 450 sophomores, 345 juniors, and 205 seniors in your school. The student council has 25 delegates. Use the Adams method for apportioning the 25 delegates. (Can you avoid trial and error by adapting the method used in Exercise 4 for this situation?)

6. Use the Webster method to apportion 50 delegates to states A, B, and C with populations 284,000, 488,000, and 228,000, respectively. (Can you adapt the method used in Exercise 4? Hint: On a TI-83, round(P_A/x, 0) will round the ratio to the nearest integer. See the MATH, NUM menu.)

What's next? Stay tuned for the Hill method, the method presently used to apportion the U.S. House of Representatives, in the next MMOW Pull-Out section.