In our everyday lives, we often hear phrases involving averages: the student has a grade-point average of 3.12, the baseball player has a batting average of .312, the average monthly rainfall for November is 3.12 inches, etc. In this HiMAP Pull-Out Section, we will investigate two types of averages and look at some of their properties, in addition to discussing an application of these properties in our daily lives.

References


The expressions, "average of the rates" and "the average rate," sound very similar but, as we shall see, they are different concepts. Consider the following situation.

Example 1:

Suppose that on a calm day, an airplane flies from city A to city B at a constant rate of $r_1$ miles per hour and on the return trip, the same airplane flies at a constant rate of $r_2$ miles per hour. Determine the average of the rates and the average rate.

Solution:

The average of the rates is the arithmetic mean of the rates:

$$\text{average of the rates } r_1 \text{ and } r_2 = \frac{r_1 + r_2}{2}. \quad (A)$$

The average rate is the total distance divided by the total time. For the round trip, $t_1$ (hours) is how long it took to go from city A to city B at the rate $r_1$, and $t_2$ (hours) is the time it took for the return trip at the rate $r_2$, then the average rate of $r_1$ and $r_2$

$$= \frac{r_1 t_1 + r_2 t_2}{t_1 + t_2}, \text{ where } r_1 t_1 = r_2 t_2$$
$$= \frac{(r_1 t_1 + r_1 t_1)/(t_1 + (r_1 t_1/r_2))}{t_2} = \frac{(2r_1 t_1)/(t_1[1 + (r_1/r_2)])}{r_2}$$
$$= 2r_1 / [(r_1 + r_2)/r_2]$$
$$= 2(r_1 r_2)/(r_1 + r_2). \quad (B)$$

Notice that the average of the rates (A) and the average rate (B) have both been expressed in terms of the two given rates, $r_1$ and $r_2$. Let’s now investigate these two concepts further.

You Try It #1

Using the two rates $r_1$ and $r_2$, develop an algebraic expression by doing the following:

- find the reciprocals of $r_1$ and $r_2$.
- find the arithmetic mean of the two reciprocals.
- find the reciprocal of the arithmetic mean.

Simplify the result and describe what you have found.

The average of the rates and the average rate, are two important mathematical topics. They are respectively called the Arithmetic Mean (AM) and the Harmonic Mean (HM) of two positive real numbers $x$ and $y$, where, by definition,

- the Arithmetic Mean of $x$ and $y = AM[x, y] = (x + y)/2$, and
- the Harmonic Mean of $x$ and $y = HM[x, y] = 2xy/(x + y)$. 

Related to Example 1, we see that

the average of \( r_1 \) and \( r_2 = \text{AM}[r_1, r_2] \), which is (A), and
the average rate of \( r_1 \) and \( r_2 = \text{HM}[r_1, r_2] \), which is (B).

From the simplified results of You Try It #1, we see that the Arithmetic Mean and the Harmonic Mean are related by the following expression:

\[ \text{HM}[r_1, r_2] = \frac{1}{\text{AM}[1/r_1, 1/r_2]} \]

Let’s now take a look at another property of these two means. Suppose that \( x \) and \( y \) are two positive real numbers, such that \( x \geq y \). Then

\[
\begin{align*}
(x - y) & \geq 0, \\
(x - y)^2 & \geq 0, \\
x^2 - 2xy + y^2 & \geq 0.
\end{align*}
\]

By adding \( 4xy \) to both sides, we get

\[
\begin{align*}
x^2 + 2xy + y^2 & \geq 4xy, \\
(x + y)(x + y) & \geq 4xy.
\end{align*}
\]

Dividing both sides by \( 2(x + y) \), which is greater than 0 (why?), we get

\[
\frac{x + y}{2} \geq \frac{2xy}{x + y}.
\]

We have assumed above that \( x \geq y \); if \( y \geq x \), one obtains the same results. Hence, if \( x \) and \( y \) are two positive real numbers, then

\[ \text{AM}[x, y] \geq \text{HM}[x, y]. \quad \text{(C)} \]

You Try It #2

At the beginning of the school year, your mathematics teacher tells you that during the school year, you will have only two tests to determine your grade for the course and that you can decide how those two scores are to be averaged. If your two choices are \( \text{AM}[\text{test}_1, \text{test}_2] \) of \( \text{HM}[\text{test}_1, \text{test}_2] \), which method would you select? When would these two means be the same?

You Try It #3

Find the average of the rates \( r_1 \) and \( r_2 \) and the average rate of \( r_1 \) and \( r_2 \) for a round trip in an airplane on a calm day if you travel half of the time at rate \( r_1 \) and the other half at rate \( r_2 \).
Example 2:

Up until this point in the discussion, the airplane has been flying on a calm day. Let’s now assume there is a wind blowing at a constant velocity of \( r \) miles per hour that acts as a head wind on the way to city B and as a tail wind on the return trip Find the average of the rates and the average rate for a round trip.

Solution:

The average of the rates \((r_1 - r)\) and \((r_2 + r)\) is

\[
AM[(r_1 - r), (r_2 + r)] = \frac{(r_1 - r) + (r_2 + r)}{2},
\]

\[
= (r_1 + r_2)/2, \text{ which } = AM[r_1, r_2].
\]

This tells us that the rate of the wind, \( r \), has no effect on the average of the rates \((r_1 - r)\) and \((r_2 + r)\).

When we consider the average rate of \((r_1 - r)\) and \((r_2 + r)\), it is

\[
= \frac{(r_1 - r)t_1 + (r_2 + r)t_2}{(t_1 + t_2)}, \text{ where } (r_1 - r)t_1 = (r_2 + r)t_2,
\]

\[
= \frac{2(r_2 + r)t_2}{(r_1 - r)t_1 + (r_2 + r)t_2}, \text{ where } t_2 = (r_1 - r)t_1/(r_2 + r), \text{ which becomes}
\]

\[
= \frac{2(r_1 - r)(r_2 + r)}{(r_1 - r)(r_2 + r)},
\]

\[
= HM[(r_1 - r), (r_2 + r)].
\]

From this we can see that the rate of the wind, \( r \), does have an effect on the average rate of \((r_1 - r)\) and \((r_2 + r)\).

You Try It #4

In You Try It #3, we determined the average of the rates and the average rate for a round trip in an airplane on a calm day if you travelled half the time at rate \( r_1 \) and the other half the time at rate \( r_2 \). Now determine the average of the rates and the average rate for a round trip on a windy day if half of the time is travelled at the rate \((r_1 - r)\) and the other half travelled at the rate \((r_2 + r)\).

Before we go any further, let’s summarize some of our findings so far when two rates, \( r_1 \) and \( r_2 \) are specified.

<table>
<thead>
<tr>
<th>Example 1</th>
<th>You Try It #3</th>
<th>Example 2</th>
<th>You Try It #4</th>
</tr>
</thead>
<tbody>
<tr>
<td>calm day</td>
<td>calm day</td>
<td>windy day</td>
<td>windy day</td>
</tr>
<tr>
<td>equal dist.</td>
<td>equal time</td>
<td>equal dist.</td>
<td>equal time</td>
</tr>
<tr>
<td>Average of the Rates</td>
<td>( AM[r_1, r_2] )</td>
<td>( AM[r_1, r_2] )</td>
<td>( AM[r_1 - r, r_2 + r] )</td>
</tr>
<tr>
<td>Average Rate</td>
<td>( HM[r_1, r_2] )</td>
<td>( AM[r_1, r_2] )</td>
<td>( HM[r_1 - r, r_2 + r] )</td>
</tr>
</tbody>
</table>

So far, we have looked at the average of rates and the average rate for various assumptions of distance travelled, time travelled, and the type of day – calm or windy. Let’s now compare the amount of time necessary to make a round trip between cities A and B on a calm day and on a windy day.
Example 3:

In Example 1, the round trip was made on a calm day while in Example 2, the round trip was made on a windy day. Now let's compare the total time travelled during the round trip on a windy day with that for a round trip on a calm day, where we assume that $r_1 = r_2$. We will also assume that $r_1 > r$. (Why is this a reasonable assumption and where is that assumption used below?)

Solution:

Total time on a windy day \\
$= \frac{d}{(r_1 - r)} + \frac{d}{(r_1 + r)}$, where $d$ is the one–way distance, \\
$= \frac{d}{(r_1 - r)} + \frac{d}{(r_1 + r)}$, since $r_1 = r_2$, \\
$= \frac{d(r_1 + r_1 - r)}{(r_1 - r)(r_1 + r)}$, \\
$= \frac{d(r_1 + r)}{(r_1 - r)(r_1 + r)}$, \\
$= \frac{2d}{(r_1 - r)(r_1 + r)} + \frac{2d}{(r_1 + r)}$, \\
$= \frac{2d}{(r_1 + r)}$. \\

Note that this is the total distance, $2d$, divided by the average rate on a windy day. (See the table on page 4.) Since \\
$AM[(r_1 - r), (r_1 + r)] \geq HM[(r_1 - r), (r_1 + r)]$, or \\
$1/HM[(r_1 - r), (r_1 + r)] \geq 1/AM[(r_1 - r), (r_1 + r)]$, \\
then the total time on a windy day $\geq \frac{2d}{[(r_1 - r) + (r_1 + r)]/2} = \frac{2d}{r_1}$, \\
which is the total time on a calm day.

Surprise! This tells us that the total round–trip time on a windy day is longer than the total time on a calm day if $r_1 = r_2$ and $r_1 > r$. Did you expect this result?

So far, we have investigated the Arithmetic Mean and the Harmonic Mean in terms of some hypothetical situations of an airplane making round trips on either a calm day or on a windy day. Now let's see how some of that carries over into our everyday lives.

When a person purchases a car, fuel economy is one of the buyer’s major concerns. Each year, the U.S. Department of Energy publishes The Gas Mileage Guide [U.S. Government 1986], which contains information about fuel–economy estimates in terms of miles per gallon of gasoline. These fuel–economy estimates are based on results of the U.S. Environmental Protection Agency (EPA) official emission and fuel–economy test procedures. The miles–per–gallon figures for city and highway driving that are listed in The Gas Mileage Guide are very useful when comparing vehicles. The average estimates for city driving and for highway driving are also attached to the window of each new automobile.

Example 4:

For the 1987 Toyota Corolla with a five–speed manual transmission, the EPA estimates that one should obtain 30 miles per gallon for city
driving and 37 miles per gallon for highway driving. Find the Arithmetic Mean and Harmonic Mean of these two rates.

Solution:

\[ \text{AM}[30, 37] = \frac{30 + 37}{2} = 33.5 \text{ mpg}, \text{ and} \]
\[ \text{HM}[30, 37] = \frac{2(30)(37)}{30 + 37} = 33.13 \text{ mpg}. \]

In the 1987 *Gas Mileage Guide*, one reads that "the reader is cautioned that simply averaging the mpg for city and highway driving ... will yield an incorrect answer (when one determines an annual fuel cost)." Explain why neither the Arithmetic Mean nor the Harmonic Mean of the city mpg and the highway mpg should be used to determine an average of miles per gallon.

Returning to the 1987 Toyota Corolla in Example 4, suppose that you purchase one and drive it for 22,000 miles in one year, of which 9,000 miles were city driving and the remaining 13,000 miles were highway driving. With this information, we can determine the car’s mpg, as follows.

\[
\text{mpg} = \frac{\text{miles of city driving} + \text{miles of highway driving}}{\text{gals. for city driving} + \text{gals. for highway driving}}
\]

\[
= \frac{(9,000 + 13,000)/[(9,000/30) + (13,000/37)]}{},
\]
\[= 33.78 \text{ mpg}.
\]

If you had driven 13,000 in the city and 9,000 miles on the highway, then

\[
\text{mpg} = \frac{(13,000 + 9,000)/[(13,000/30) + (9,000/37)]}{},
\]
\[= 32.52.
\]

Both of these means are called "weighted harmonic means" [Wagner 1981, 3–4] and can be generalized in the following manner. Suppose that a car is driven annually \(m_c\) miles in the city at a rate of \(c\) miles per gallon and \(m_h\) miles on the highway at a rate of \(h\) miles per gallon. Then

\[
\text{mpg} = \frac{(m_c + m_h)}{(m_c/c + m_h/h)}
\]
\[= 1/[(1/c)(m_c/(m_c + m_h)) + (1/h)(m_h/(m_c + m_h))].
\]

If we let \(i = m_c/(m_c + m_h)\), then \(i\) is the percentage of miles driven in the city, and if we let \(j = m_h/(m_c + m_h)\), then \(j\) is the percentage of miles driven on the highway. Hence,

\[
\text{mpg} = 1/(i/c = j/h),
\]
\[(D)\]

where \(i\) is the percentage of city–driven miles at \(c\) miles per gallon and \(j\) is the percentage of highway–driven miles at \(h\) miles per gallon.
If \( i = j \) in (D), determine what the mpg becomes.

Returning to Example 4 for the 1987 Toyota Corolla, suppose that you owned one and you drove it in the city 55 percent of the time and on the highway 45 percent of the time. Find the following:

a. the number of miles per gallon using (D).

b. the cost of driving the car 20,000 miles if gasoline costs $0.89 9/10 per gallon.

C. Conclusion

We have just seen that the Harmonic Mean and the Weighted Harmonic Mean can be used in everyday life. These averages, along with the Arithmetic Mean, play an integral part in our lives. Use them often!

Solutions to the "You Try Its"

1. \( 1/[(1/r_1 + 1/r_2)/2] = 2/[1/r_1 + 1/r_2] = (2r_1r_2)/(r_1 + r_2) \), which is the average rate of \( r_1 \) and \( r_2 \).

2. Since \( \text{AM}[x, y] \geq \text{HM}[x, y] \), I would select \( \text{AM}[\text{test}_1, \text{test}_2] \). The two means would be the same when \( \text{test}_1 = \text{test}_2 \).

3. Average rate of \( r_1 \) and \( r_2 = (r_1t_1 + r_2t_2)/(t_1 + t_2) \), where \( t_1 = t_2 = (r_1t_1 + r_2t_1)/(t_1 + t_1) = t_1(r_1 + r_2)/(2t_1) = (r_1 + r_2)/2 \), which is the average of the rates, \( \text{AM}[r_1, r_2] \).

4. Average rate = \( [(r_1 - r)t_1 + (r_2 + r)t_2]/(t_1 + t_2) \), where \( t_1 = t_2 = [(r_1 - r)t_1 + (r_2 + r)t_1]/(t_1 + t_1) = [(r_1 - r) + (r_2 + r)]/2 \), which is the average of the rates \( (r_1 - r) \) and \( (r_2 + r) \), or \( \text{AM}[r_1, r_2] \).

5. It is highly unlikely that a person would drive the same number of miles in the city as on the highway.

6. If \( i = j \), then \( i = j = 50\% = 0.5 \), so that \( \text{mpg} = 1/(0.5/c + 0.5/h) \), which upon simplifying becomes \( \text{HM}[c, h] \). Note that the likelihood of \( i = j \) has been discussed in exercise 5.

7a. \( \text{mpg} = 1/(0.55/30 + 0.45/37) = 32.79 \).

b. \( (20,000/32.79)(\$0.89) = \$548.34 \).