Derivations with Recursive Equations in Financial Mathematics

Floyd Vest (Preliminary Version) June 2015

Most of the above formulas in this course can be derived with Recursive Equations. As Example 1, we will derive the formula for the future value of an ordinary annuity by using Recursive Equations.

**Example 1.**

<table>
<thead>
<tr>
<th>n</th>
<th>n-1</th>
<th>n-2</th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

| R | R   | R   | R | R |

By definition:

1. \( FV_n = R(1+i)^{n-1} + R(1+i)^{n-2} + \ldots + R(1+i) + R \). The \( R \) is often read as “Rent” and is a generic term for many different applications.

2. We need to show that \( FV_n = R \left( \frac{(1+i)^n - 1}{i} \right) \).

We will let \( r = 1+i \). \( FV_n = R r^{n-1} + R r^{n-2} + \ldots + R R + R \). We will use \( P(n-1) = R r^{n-1} \), \( P(n-2) = R r^{n-2} \), \( \ldots \), \( P(1) = R r^1 \), \( P(0) = R \) and follow the index numbers in the top row. The \( P(1) \) could be read “\( P \) number 1,” “\( P \) sub 1," or “\( P \) 1.” Using this notation \( FV_n = P(n-1) + P(n-2) + \ldots + P(1) + P(0) \).

We will let \( FV_i = R \).

\[
FV_2 = FV + P(2) = R + Rr = R(1+r) = \frac{R(1+r)(1-r)}{1-r} = \frac{R(1-r^2)}{1-r}
\]

\[
FV_3 = FV_2 + P(3) = \frac{R(1-r^2)}{1-r} + R(r^2) = \frac{R(1-r^2)}{1-r} + \frac{R r^2 (1-r)}{1-r} = R \left( \frac{1-r^2 + r^2 - r^3}{1-r} \right) = R \left( \frac{1-r^3}{1-r} \right).
\]

We see the pattern. Let’s try

\[
FV_n = FV_{n-1} + P(n) = FV_{n-1} + R r^{n-1} = R \left[ \frac{1-r^{n-1}}{1-r} \right] + \frac{R r^{n-1} (1-r)}{1-r} = R \left[ \frac{1-r^n}{1-r} \right].
\]

We remember that \( r = 1+i \) giving \( FV_n = R \left( \frac{1-(1+i)^n}{1-(1+i)} \right) = R \left( \frac{1-(1+i)^n}{-i} \right) = R \left( \frac{(1+i)^n - 1}{i} \right) \) as was to be shown.

The recursive equation for \( FV \) was \( FV_n = FV_{n-1} + P(n) \).

**Example 2.**
We will derive the formula for the present value of an ordinary annuity:

\begin{equation}
(3) \quad PV_n = R \left[ \frac{1 - (1+i)^{-n}}{i} \right]
\end{equation}

by using recursive notation. Recalling the number line, by definition,

\begin{equation}
(4) \quad PV_n = R(1+i)^{-1} + R(1+i)^{-2} + \ldots + R(1+i)^{-n}.
\end{equation}

We will let \( r = (1+i)^{-1} \) giving

\[ PV_n = Rr + Rr^2 + \ldots + Rr^n = P(1) + P(2) + \ldots + P(n). \]

Deriving: \( PV_1 = Rr. \)

\[ PV_2 = PV_1 + P(2) = Rr + Rr^2 = Rr(1+r) = \frac{Rr(1+r)(1-r)}{1-r} = \frac{Rr(1-r^2)}{1-r}. \]

\[ PV_3 = PV_2 + P(3) = \frac{Rr(1-r^2)}{1-r} + \frac{Rr^3(1-r)}{1-r} = Rr \left[ \frac{1-r^3}{1-r} \right]. \]

We see the pattern:

\[ PV_n = PV_{n-1} + P(n) = Rr \left[ \frac{1-r^n}{1-r} \right] + \frac{Rr^n(1-r)}{1-r} = \frac{Rr(1-r^n + r^{n-1} - r^{n-1})}{1-r} = \frac{R(1-r^n)}{1-r}. \]

We had \( r = \frac{1}{1+i}, \) giving

\[ PV_n = R \left[ \frac{\frac{1}{1+i} \left( \frac{1}{1+i} \right) \left( 1 - \frac{1}{(1+i)^n} \right)}{\frac{1}{1+i} - \frac{1}{1+i}} \right] \frac{1+i}{1+i} = R \left[ \frac{1-(1+i)^{-n}}{1+i} \right] \]

as was to be shown.

The recursive formula is \( PV_n = PV_{n-1} + P(n) \) with \( P(n) \) defined here.

See “Mathematics Modeling Our World” at COMAP for extensive use of recursive equations.

**Example 3.** Generating the sequence \( FV_n, n = 1, 2, 3, \ldots \) in the TI83 in Sequence Mode.

Select Mode, set on Sequence. Press Y= . Complete nMin=1, u(n)=u(n-1)+100(1.10) \( \wedge \) (n-1), u (nMin)={100}. To enter a script n press [x,T,θ,n]. To enter script u, press 2nd U. You should see the above. Look at TABLE. You should see \( n \) 1 2 3 … and \( u(n) \) 100 220 331 … .

You Try It: Check and interpret these calculations. You could graph this table. (See in the Manual the Chapter “Defining and Displaying Sequence Graphs.”) Press WINDOW and make entries for Sequence graphing. Press TRACE for reading different values of \( FV_n \) and \( n. \)
**Exercises:** You may want to use multiplication in recursive equations where

\[ C(\text{next}) = C(\text{Present}) \times a, \text{ where } a \text{ is defined.} \] Show your work. Define variables and answers. Discuss in complete sentences.

#1. Derive by recursive equations the compound interest formula.

#2. Derive the formula for a sum of a geometric sequence by recursive equations:

\[ a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(1-r^n)}{1-r}. \]

#3. State and derive by recursive equations the formula for the future value of an annuity due.

You may want to derive it a second way. For an annuity due, the first of \( n \) rents \( R \) occurs at the beginning of the first period.

#4. State and derive by recursive equations the formula for the present value of an annuity due. Derive it another way.

#5. For \( I(n) \) = interest earned on interest by end of year \( n \) for compound interest, calculate \( I(n) \), calculate \( I(n-1) \), and show that the recursive formula is \( I(n) = (1+i)I(n-1)+(n-1)P\text{r}^2. \)

#6. Peter Katt has studied whole life policies paying Cash Value for many years. He reviews their history as: Participating Whole Life and Universal Life, (1977 – 1992), Variable Universal Life (1993 – 2003), Guaranteed Universal Life (2004 – present), and the latest, Index Universal Life. He says that they are all funded by premiums, and bonds and similar investments; not stocks. Insurance companies make money off of people who drop their policies. It is reported that 75% drop their policy. In spite of this, Cash Value return for Index Universal Life is claimed to be indexed to stocks. Although these policies are claimed to earn 8% or better, Katt says that 4.0% is probably needed to provide a realistic view future earnings, usually with significant increase in premiums.

White Coat developed a 4% Crediting Rate Table with $5500 annual premiums beginning at the first of the first year. The first ten years had negative returns. By the end of the fifteenth year, the return was 2.29%. What was the Cash Value? By the end of the twentieth year the cash value was $153,083. What was the sum of premiums and the rate of return. It took until after the tenth year to break even. Look up White Coat’s table and fit a curve to interpolate and find the break even point. If you get a fraction of a year, how should this be interpreted? Check and report on his calculations. Describe your methodology.


#7. See the article in this course “The Mathematics of Life Insurance Decisions,” which compares Term insurance and Whole Life. Do the research and write a similar report for Index Universal Life. You could compare with Nonparticipating Whole Life, or a long-term Term policy since after years of accumulating retirement assets, you may no longer need life insurance. Write a report of the details of the Index Universal Life policy. If you want to compare with White Coat’s table, it is for a healthy 30 year old male.
#8. People invest in Guaranteed Insurance Products for safety. Read and report on the losses in investing in stocks. You could see the articles in this course on Stock Investments. For example in the decade of the 1930s, stocks earned -0.1%, in the 2000s, -0.9% (Money, Sept. 2010). Beginning with the 1920s, what percent of the decades had a negative return? What does this tell the investor? Compare this to White Coat’s figures.

To look at stocks another way, of all the rolling 15-year calendar years beginning in 1926 and ending in 2014, the S&P has never had a negative total return. What about ten year periods? But investors are influenced by more recent history where the S&P dropped in 2000, regained in 2007, dropped again and regained in 2012 (a period of about twelve years). From Oct. 1999 to Sept. 2014 it delivered more than a 55% cumulative return, about 3% a year. The average dividends are about 2%. (AAII Journal, June 2015, page 35). For the graph, see finance.yahoo.com, S&P500 (GSPC). This 15 year recent history is not encouraging for stocks. From 1926 to 2014, the S&P 500 total return averaged 10.26% (See www.moneychimp.com/features_cagr.htm for a calculator). From Jan. 1, 1999 to Dec. 31, 2014, the moneychimp calculator gives Total Return = 5.19%.

#9. The dollar is up 24% against major currencies (Kiplinger’s Personal Finance, 6/2015, page 59). Discuss the implications for different kinds of businesses, the stock market, travelers, and so on. Do some of the simple mathematics to make price and cost comparisons.

#10. Kiplinger’s Personal Finance, 6/2015, page 60, reports on a bond fund that has a negative 3.8 years average duration. How could it have a negative duration? See the articles in this course on bond duration.

#11. Discuss the advantages and disadvantages of different kinds of derivation and proof, including: recursive equations, mathematical induction, and differences of extended sums.

#12. Do an example of generating the sequence $P_{V_n}$, $n = 1, 2, 3, \ldots$ in the TI83 in Sequence Mode. Give the code, instructions, the input, and the TABLE. Check and interpret the calculations.

#13. Do a calculation of how much life insurance is needed. Itemize expenses. Give the age, dependents, and stage in life. Consider inflation, and Social Security Survivor Benefits. Do you want to pay your kids way through college or is it alright for them to work their way. Twelve percent of college students work more than three years of high school and thirteen years of college. He is still working at the age of 81, for the fun of it. This calculation takes only common sense, and financial mathematics, but if you need help, see the internet.

#14. Scott Burns gives the following example of refinancing a mortgage: George and Sally live in a $250,000 house (not recommended). The mortgage balance is $150,000 and the mortgage rate is 5.5 percent. This year they expect to pay about $8250 in interest. They refinance their home at 4.5 percent, which cuts their interest by $1500 a year. The refinancing costs $3000. Check Scott’s figures. Do any other calculations you want. How much money at short term interest rate would earn $1500 a year? What rate of interest are they making on the $3000 in expenses. Would their payments be smaller? (Denton Record Chronicle, Oct. 10, 2010, page 2D)
**Side Bar Notes.** See articles in this course:

**Money worries:** A released survey by the American Psychological Association found that money is the top source of stress for the majority of Americans, above work, family, and health. What about a course in financial mathematics? (Money, July 2015, p. 9)

**The most hated fees in America.** Consumers pay about $32 billion a year in overdraft charges. Can you teach common sense? (Money, July 2015, page 18)

**Two-thirds of workers** file for Social Security before full retirement, age 66, and only about 2% wait until age 70 when benefits max out. (Money, June 2015, page 33)

**“Half-Life of Facts:** From “Why Everything We Know Has an Expiration Date,” by the mathematician Samuel Arbesman, many facts will be shown to be wrong in less than five years. Pure mathematic and much of applied mathematics remains always true. Each body of knowledge has its own definition of “truth.” (Denton Record Chronicle, June 9, 2015)

**Dividend growers** in the S&P 500 for the last five years have earned 10.1%, Dividend Payers 9.3%, S&P 500 7.8%, Nonpayers 2.6%. (Money, July 2015, page 66)

Although Index Universal Life policies claim to be indexed to the stock market, they do not include dividends.

**Largest U.S. asset managers.** BlackRock $4.7 trillion, Vanguard $3.1 trillion, State Street $2.5 trillion, Fidelity $2.1 trillion. Index investing is the big deal. These managers own such large portions of certain companies’ stocks that they can dictate management policy.

**College degree, marriage, and children.** Among college educated, relatively affluent couples, marriage is doing pretty well. Where education and income levels are lower, it’s often a different story. College educated are more likely to wait until marriage to have children and then raising them supported by two incomes. Unmarried mothers account for 40.6 percent of births. For African Americans, the rate is 71.5 percent. What does this say about an obligation to acquire education and earning power?

**Lifespan at age 65 by educational level.** For Women: High school, age 84; College, age 87; For Men: High School, age 81; College, age 84. (Money, July 2014) Discuss the many reasons why education might be associated with lifespan.

**Stock bubbles happen.** On March 10, 2000, the Nasdaq Composite Index peaked at 5048. Then it fell, and fell. Finally down to 1114. The bubble burst. As judged by price-to-earnings ratios many stocks were grossly overvalued with examples of Cisco Systems (CSCO) at P/E of 148, Oracle (ORCL) at P/E of 153, Qualcomm (QCOM) 167. Other indexes including the S&P 500 plunged, but recovered. The Nasdaq by contrast was still 7% below its former high as of Jan. 9, 2015. The promised revolution of the internet failed stock prices. Check these figures on finance.yahoo and look at the P/Es of the S&P 500. What are reasonable P/Es for the S&P 500? (Kiplinger’s Personal Finance, 03/2015, page 18)
The “unreasonable effectiveness of mathematics in describing the physical universe” is a phrase composed by the great mathematical physicist Eugene Wigner. This phenomenon has been confirmed numerous times in the history of mathematics and science. It was once again confirmed when Peter Higgs displayed his famous equation describing the source of a particle’s mass, the Higgs boson field. It took half a century for it to be verified. (Smithsonian.com, July, August 2013, page 24) Much of applied mathematics was first discovered as pure mathematics without any apparent application. Pure mathematics is often jokingly referred to as applicable mathematics.

Long term care such as in a nursing home is an important consideration in investing for retirement. Beginning in 2010, the government allows consumers to tap the death benefit of their whole or universal life insurance, or the balance in an annuity to cover long-term-care-expenses, tax free. (Smart Money, Feb. 2011) Would this be enough? As an excise, calculate the accumulated cost of ten years in a nursing home beginning when you are 80 years old. Consider current costs, and inflation in nursing home costs, including yearly inflation of costs during the ten years. In 2012, the average cost was $76,285. You will be amazed at how much it costs. People often celebrate their 10th anniversary in a nursing home and have enjoyed the nursing home environment. They even celebrate their 100th birthday. But if you run of money, Medicaid will pay. Some resources and income are left for a surviving spouse.

Scott Burns reports that for women, at age 75, only 12.4% will need care for more than two years. Only 3.9% will need care for more than five years. (Denton Record Chronicle, Dec. 2, 2012, page 4D)

Solvent Seniors. Forty eight percent of married seniors and 28 percent of single seniors have more income from investments than from Social Security. Can they live comfortably on this total income? (Scott Burns, Denton Record Chronicle, March 21, 2010) Do the calculations: How much do you need in investments at retirement to match this? This is a common calculation. See the articles in this course on “Long Term Financial Planning”, and do the research on your Social Security. If this is too much trouble, do some estimates. How much a year do you need to invest?

Understanding Social Security Credits. Of your income, the first $767 a month, is credited at 90%,.90(767) = $690 a month in Social Security, wages more than $767 a month and up to $4624 a month are credited at 32%. Wages from $4624 a month up to $113,700 a month are credited at 15%. Lowest wage earners can expect Social security benefits nearly equal to their income. Wages are figured as average indexed monthly earnings (a complicated formula that adjusts earnings over time). (Scott Burns, Denton Record Chronicle Jan. 13, 2013)

Have you wondered about reverse mortgages? Scott Burns (Denton Record Chronicle, June 3, 2012) gives the following example he got from an online calculator. The reverse mortgage on a $250,000 house, which is paid for, would cost $11,626 in fees, a net of $147,624 after fees. For an age 65 couple, a tax free monthly income of $822 as long as they live in the house is an option. Over 20 years the amount owed (principal advanced plus interest) will have compounded to about $380,000. After 20 years of this reverse mortgage income, will there be any value of the house left? It depends on the appreciation (inflation) rate. At 3%, the house is worth about $450,000. At 2%, what is it worth?
The $822 a month is about $9864 a year. They still pay insurance, taxes, upkeep, and utilities. Maybe they have Social Security of over $20,000 a year.

For an age 65 couple, a life annuity paying $822 a month (which may be partly taxable) costs $165,000. If they sell the house for $250,000 less expenses, they might get $225,000. This leaves $60,000 after paying the annuity.

For this couple, if they deposit $200,000 in a Variable Annuity with a Life Benefit, it would pay about $822 a month. They would have about $25,000 left from the sale of the house.

What about selling the house and investing the $225,000? The author’s advice: If only $9864 a year helps that much, both of them should get a job, and they should cut their housing expenses.

If over 20 years, the amount they owed (principal advanced plus interest) on $822 at the beginning of each month is $380,000, what is the annual effective rate of interest being charged?

Assignment: What kind of couple would own a $250,000 house? Report on the options, making certain assumptions, and considering flexibility, personal preferences and needs, and using financial mathematics. The average annual energy bill for a typical single family home is $2200. By how much could this be reduced? On average, during retirement, virtually every category of expenses declines.

One major U.S. bank got out of the reverse mortgage business because so many customers violated their contract by such things as not paying their property taxes. The bank’s excuse was that they didn’t want the bad PR of foreclosing on destitute seniors.

Only 16% of college graduates did degrees in STEM (science, technology, engineering or mathematics). They made 21% more than other graduates. Many of them are foreign nationals or second generation foreign descent. Half the honors students are of the same background. Computer science made 47% more. Asians made more than others. Graduates from for-profit colleges made 24% more than others. Petroleum Engineers made about $120,000. The average for non-science degrees was $49,500. The average unemployment rate for graduates was 6.7% while 8.1% was the national unemployment rate. Graduates of Black Colleges have an unemployment rate of 12%. (Denton Record Chronicle July 9, 2014, The Associated Press).

For Millennials, born after 1960, 34% have 4-year degrees while for previous generations, 25% have degrees. But among young adults, 36% are living with parents.

Advantages of the Roth IRA and Roth 401k over the Traditional IRA and 401k. The maximum contribution for both the Roth IRA and the Traditional IRA is $5500. On a tax equivalent, the Roth allows a larger contribution. The limit on the traditional IRA of $5500 (before taxes) is tax equivalent at the 25% bracket to 5500(1-.25) = $4125. But, the Roth allows $5500 after taxes which is greater on a tax equivalent basis than the $4125. So on a tax equivalent basis, the Roth allows a larger contribution. As a result, the Roth will accumulate more after tax funds at retirement than the Traditional. Looked at another way, at the 25% tax bracket, a $5500 after tax contributed to a Roth IRA is equivalent to B with B(1-.25) = 5500.

\[ B = 7333.33 \] before taxes while the limit on the Traditional IRA is $5500 before taxes.
However, other things being equal, if the amounts contributed to the Roth and Traditional are equal in terms of tax equivalency, they will yield the same accumulation after taxes. (See the article in this course, “Does T M Need a Roth?”)

Another advantage of the Roth is withdrawals from a Traditional IRA or 401k do count against you when it comes to determining the portion of Social Security benefits which is taxed. Withdrawals from the Roth do not usually count. Withdrawals from the Traditional can increase the portion of Social Security that is taxed, withdrawals from the Roth will not increase the portion that is taxed. This is an advantage for a limited range of income. If none of your Social Security is taxed, or if the maximum 85% of your Social Security is already taxed, this is not an advantage. (See instructions for IRS 1040, the Worksheet for Taxation of Social Security Benefits, and Form 8606. See finance.zacks.com or www.fool and search. See the article in this course, “Mathematical Induction in Financial Mathematics,” Exercise #17. See Money, Aug. 2014, page 37, “Roth 401k,. . . Good News if You Want More Retirement Income.”

Taxes and tax rates can increase during retirement because of Mandatory Withdrawals on Traditional IRAs.

References:

A free course in financial mathematics, with emphasis on personal finance, for upper high school and undergraduate college, at COMAP.com. Register and they will e-mail you a password. Simply click on an article in the annotated bibliography, download it, and teach it or study it. Unit 1: The Basics of Mathematics of Finance, Unit 2: Managing Your Money, Unit 3: Long-Term Financial Planning, Unit 4: Investing in Bonds and Stocks, Unit 5: Investing in Real Estate, Unit 6: Solving Financial Formulas for Interest Rate. The last section is Additional Articles on Financial Mathematics or Related to Personal Finance. In all, there are about seventy articles. For about thirteen more advanced or technical articles, see the UMAP Journal at COMAP.

Teachers’ Notes:

For more applications of recursive equations in financial mathematics, in lessons at an elementary level, on compound interest, and decay in purchasing power of the dollar, where students learn modeling skills, see at COMAP.com the publication: Tech MAP “Tim and Tom’s Financial Adventure” which is a supplement to “Course 1, Mathematics: Modeling Our World.”

See moneychimp.com for dozens of instructions, investment concepts, calculators, and financial mathematics concepts, for investing. There is a whole course in investing in this site. How much of the concepts and mathematics in this site is explained in this course?

You can find lessons on recursive equations on the internet, and a more advanced treatment at Wikipedia.org.