In business transactions, it is often necessary to meet a prearranged obligation but at a different time than that originally agreed upon.

**Example 1. Compound Interest.** A person owes $20,000 due in one year and $30,000 due in three years at 5%. The borrower has the agreement to pay the debt now and the creditor agrees to 5%. We can put this original transaction on a time line with “now” at Time zero.

\[
\begin{array}{cccccc}
\text{0} & \text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} \\
\$20,000 & \$30,000 & & & & & \\
\end{array}
\]

By the principle of discounting compound interest, the amount owed at time zero is \( x \) (1) where \( x = 20,000(1 + .05)^{-1} + 30,000(1 + .05)^{-3} \) and \( x = 44,962.75 \). This equation is called an Equation of Value with Focal Point at Time 0, and represents the original agreement.

But things happen. The borrower gets permission to pay off at the end of the first year. How much is owed at that time? To answer this question we can move the Focal Point forward by one year putting the Focal Point Year 1 at the $20,000, by multiplying Equation 1 through by \((1 + .05)^1\) to get \( x(1 + .05)^1 \) at the new Focal Point.

\[
\begin{array}{cccccc}
\text{0} & \text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} \\
\$20,000 & \$30,000 & & & & & \\
\end{array}
\]

(2) \( x(1 + .05)^1 = 20,000 + 30,000(1 + .05)^{-2} \). Solving gives \( x(1 + .05)^1 = 47,210.89 \) a the pay-off at the new Focal Point at end of Year 1.

So we have two equivalent Equations of Value, Equation 1 with the Focal Point at Time 0 and Equation 2 with Focal Point at Year 1. Equivalent equations are equations with the same solutions. But we didn’t solve, at the last, for \( x \) but for \( x(1+.05) \).

We can show that Equation 2 makes sense by setting aside in savings \( x = 44,962.75 \) at time zero. After one year at 5%, at Focal Point 1 Year, there is 44,962.75(1 + .05) = 47,210.88 which pays the $20,000 leaving $27,210.88. Investing this for two more years gives 27,210.88(1 + .05)^2 = $30,000, as it should be to pay off the debt at the end of the third year.

A way to look at Equation 1 (the Principle of Discounting) is to consider a savings of \( x = x_1 + x_2 \) in dollars at Time 0 where \( x_1(1 + .05)^1 = 20,000 \) so
\[ x_1 = 20,000(1 + .05)^{-1}, \text{ and } x_2(1 + .05)^3 = 30,000 \text{ so that } x_2 = 30,000(1 + .05)^{-3}. \] This give \[ x = 20,000(1 + .05)^{-1} + 30,000(1 + .05)^{-3} \] as in Equation 1 with Focal Point at Time 0.

As we see, different equivalent Equations of Value for an equivalent time value of money can represent different scenarios. Consider Example 2.

Example 2. We have here a different kind of story. A scenario about savings and withdrawals, where $700 in savings at Time 0 in the bank provides for a withdrawal of $300 at the end of Year 2 and \( x \) dollars at the end of Year 10. On the time line below, we put the Focal Point at Time 0.

\[
\begin{array}{ccccccccccccc}
\text{\$700} & | & \text{\$300} & | & \text{$x$} \\
0 & | & 1 & | & 2 & | & 3 & | & 4 & | & 5 & | & 6 & | & 7 & | & 8 & | & 9 & | & 10 & | & 11 \\
\end{array}
\]

Let \( x \) = the amount of the withdrawal at end of Year 10, and $700 at the Focal Point at Time 0, and $300 withdrawn at Time 2. By the principle of discounting $700 = 300(1 + i)^{-2} + x(1 + i)^{-10}$. To solve for \( x \), we can move the Focal Point to Year 10 by multiplying through by \((1 + i)^{10}\) to get $700(1 + i)^{10} = 300(1 + i)^8 + x$, and solve for \( x \). If \( i = .05 \), we get the amount withdrawn at Year 10 of $696.99. In this example, we have a different Equation of Value for purpose of solving for \( x \).

Example 3. It is agreed that a loan of \( P \) dollars at Time 0 is to be paid off in two equal payments of \( x \) dollars at the end of Year 1 and Year 2. By the principle of discounting, \( P = x(1 + i)^{-1} + x(1 + i)^{-2} \). Solving for \( x \) gives \( P = x\left[(1 + i)^{-1} + (1 + i)^{-2}\right] \) and \( x = \frac{P}{(1 + i)^{-1} + (1 + i)^{-2}} \). We can move the Focal Point to the end of Year 2 by multiplying through by \((1 + i)^2\) to get an equivalent Equation of Value of \( P(1 + i)^2 = x(1 + i)^3 + x \). Solving for \( x \) gives \( P(1 + i)^2 = x[1 + i + 1] \) and \( x = \frac{P(1 + i)^2}{2 + i} \). We know by the time value of money and by algebra that the two expressions for \( x \) of \( x = \frac{P}{(1 + i)^{-1} + (1 + i)^{-2}} \) and \( x = \frac{P(1 + i)^2}{2 + i} \) are equal.

At this stage, it should be clear that a sum has different time values. For example $100 on hand is worth more than $100 two years on the future. If 8% effective is applied, the one hundred dollars payable today requires more $100(1 + .08)^2 = 166.64 due two years from today. So $100 today pays all of the debt, but in two years pays only part of it.

To compare sums of money at different times, you pick some Focal Point and find the sums by principal and interest at that date. To do this there must be an agreement.
between the two investors as to the timing and prevailing interest rate. Shifting the Focal Point may or may not represent a different scenario. An Equation of Value can represent a mathematical manipulation, a lender and borrower scenario, a savings scenario, a withdrawal scenario, or an agreement between two investors. The equations can even represent phenomena in nature. The mathematical equation tells only part of the story, but it is a mathematical model of the story. It is the abstract and general nature of mathematics that makes it powerful. By the way, tell your banker that it is only a simple matter of multiplication. For a discussion of equivalent equations, see the Side Bar Notes. See Shane and Cissell in the References or similar books for further examples of Equations of Value for Compound Interest.

**Equations of Value for Simple Interest, Not so simple.** By basic definitions of Simple Interest, \( S = P(1 + ni) \) so \( P = \frac{S}{1 + ni} \) where \( S \) is the Future Value of Sum, \( n \) is time in years, \( P \) is the Present Value or Principal, and \( i \) is the simple interest rate.

**Example 4.** Consider the following example which illustrates the Time Value of Money.

Agreement 1. You have an agreement with a lender to pay off a debt at 6% simple interest with $300 owed now and $106 owed in one year. Agreement 2: With Pay Off of the whole debt now, the Pay Off = $300 + \( \frac{106}{1 + .06} \) = 300 + 100 = $400. Agreement 3: With a Pay Off in one year, the Pay Off = 300(1 + .06) + 106 = $424.

The three alternatives can be summarized as $300 now plus $106 in one year, $400 now, or $424 in one year. All alternatives are equivalent by the Time Value of Money if money is worth 6% simple interest. (Cissell, p. 40).

**Example 5.** First Agreement: An investor borrows $2000 at 15% simple interest on June 1, 1996. The debt is to be repaid by equal installments on June 1, 1997 and June 1, 1998. Find the size of the payments. See the timeline below.

<table>
<thead>
<tr>
<th>Focal Point</th>
<th>June 1, 1996</th>
<th>June 1, 1997</th>
<th>June 1, 1998</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2000</td>
<td>$x</td>
<td>$x</td>
<td>$x</td>
</tr>
</tbody>
</table>

For the Focal Point and the first \( x \), \( P_1 = \frac{x}{1 + .15(1)} \) and for the second \( x \), \( P_2 = \frac{x}{1 + .15(2)} \). Solving for \( x \) gives

\[
P = P_1 + P_2 = 2000 = \frac{x}{1 + .15(1)} + \frac{x}{1 + .15(2)}.
\]
Second Agreement: The parties agree to a Focal Point of June 1, 1998. Setting the Focal Point at June 1, 1998 gives the equation

\[ x(1 + .15) + x = 2000(1 + .15 \times 2), \]

and

\[ x = \frac{2000(1 + .30)}{2 + .15} = $1209.30. \]

Notice that “unexpectedly” the second calculation for \( x \) is $11.11 less than the one before. But both parties have agreed to the 15% simple interest and the Focal Point.

What is the problem? The answer is that the simple interest manipulations we did on the Equation of Value do not produce equivalent equations – equations with the same solutions. For this reason it is important for both parties to agree to the new Focal Point. In changing from Equation 4 to Equation 5 we didn’t use the common manipulation of multiplying both sides of the equation by the same none zero value. Consider this possibility for Equation 4:

\[ \frac{x}{1.15} + \frac{x}{1.30} = 2000. \]

Multiplying through by 1.30 gives

\[ \frac{x(1.30)}{1.15} + x = 2000(1.30) \]

which isolates the second \( x \) as in Equation 5. Solving for \( x \) gives \( x = $1220.41 \) which is the same as the first agreed payment of $1220.41, with the Focal Point at June 1, 1996, not June 1, 1998.

One problem with Simple Interest is that: As \( n \) increases, the effective simple interest rate \( i_n \) decreases. Consider the simple interest formula for \( P = $1. \)

\[ S(n) = 1 + in \]

where \( i \) is the simple interest rate and \( n \) is the number of years, and \( S(n) \) is the Future Value or Sum for $1. (Kellison, p.5-6).

Proof: \( i_n = \frac{S(n) - S(n-1)}{S(n-1)} = \frac{[1 + in] - [1 + i(n-1)]}{1 + i(n-1)} = \frac{i}{1 + i(n-1)}. \) Notice that as \( n \) increases, \( i_n \) decreases for \( n \geq 1. \) (See the Exercises for some more results for simple interest.)

The United States Rule (Cissell, p. 55). Instead of having one Focal Point, each time a payment is made, the preceding balance is brought to interest at the time of the payment.

**Example 6.** Consider the time line showing a debt of $2000 at Present, a payment of $300 at Year 1, a payment of $200 at Year 2, and a balance due of $x at Year 3.

<table>
<thead>
<tr>
<th>Present, 0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>years</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2000</td>
<td>$300</td>
<td>$200</td>
<td>$x</td>
<td></td>
</tr>
</tbody>
</table>
Balance due

Original debt $2000
Interest on $2000 for one year at 5% 100
Balance due 2100
Deduct payment 300
New balance due 1900
Interest on $1900 for one year at 5% 95
Balance due 1995
Deduct payment 200
Balance 1795
Interest of $1795 for one year at 5% 89.75
Balance due at Year 3 1884.75

Merchant’s Rule (Cissell, p. 53)
Original debt $2000
Interest for 3 years 300

First payment 300
Interest for 2 years 30
330

Second payment 200
Interest for 1 year 10
210

Sum of partial payments 540
Balance due at Year 3 1760

The Merchant’s Rule is equivalent to putting the Focal Point at the time of the final settlement:

$$2000(1+.15) - 300(1 + .10) - 200(1 + .05) = x.$$  
$$x = 1760.$$  
When you see your merchant, ask him where the Focal Point is?

**Side Bar Notes:**

**Equivalent equations.** In a careful algebra course, by reviewing the properties of real numbers and defining equivalent equations, one can prove that multiplying an equation through by a nonzero factor produces an equation which is equivalent to the former equation.

**Simple Interest for an amortization schedule for a mortgage.** Once the periodic payment has been calculated, simple interest can be used to complete the table. See the article in this course, “The Mathematics of Amortization Schedules”.
Other forms of interest. You can study Add-On Interest, Discount Interest, Rule of 78, and so on. See books similar to the References in this article. On amazon.com and banrsandnoble.com you can buy used college textbooks for $.10, less than $5 including shipping.

**Code and comments for the TI84 Solver:** Exercise 9. On a clear home screen: Math 0 for the Solver You see Equation Solver eqn:0= Write in an equation in x for the 1+i and equal to zero. Enter You see x= Write in a first guess. Alpha Solve You should see x=1.122. This gives the interest rate.

**Money and Happiness?** A field of economics has grown up around this question, happiness economics. One equation is \( W = \alpha + \beta x + \varepsilon \). For day-to-day happiness, it improves up to an income of about $75,000 and levels off. Achievement happiness comes with fulfilling a need to achieve something – including money. Material happiness is short lived after a new acquisition (Kiplinger’s Personal Finance, 1-2012).

**College education tax breaks available (2011).** The American Opportunity Tax Credit (AOTC) and the tuition and fees deduction (tfd): The AOTC allows a tax credit up to $2500 per student. The tfd allows a deduction up to $560 per family (at 28% tax bracket). You can’t take both. Income limits apply (MONEY magazine, Dec. 2011).
Exercises: Show your work. Label, numbers, variables, and answers. Supply formulas.

1. (a) An investor deposited $6000 three years ago in an account paying 5% compounded annually. One year ago he withdrew $4000. Today (at Time 0) he withdrew $1000, in three years he will deposit an unknown amount, and at Year 12 he will close the account withdrawing $5000. Put the Focal Point at Year 3. What must the unknown deposit be? (b) Work the problem at 5% simple interest. (c) In simple interest, how much is the unknown deposit with the Focal Point at Time 0?

2. (a) Using $S(n) = P(1+ni)$ for $P$ dollars borrowed at Time 0, calculate successive Pay Offs for $n = 0, 1, 3$. (b) Show that the slope of the simple interest line $S(n) = 1 + ni$ is $i$.

3. (a) By the simple interest formula calculate the Future Value of $1 for four years at 5%. (b) Calculate the successive effective rates $i_n$ for the simple interest 5% on $P = $1 for annual periods ending in Years 1, 2, 3, and 4. (c) Show that compounding these rates give $S(4) = $1.20.

4. Consider a simple interest loan of $1, at Year 0, to be paid off in two equal payments $x$ at the end of Year 1 and Year 2, at 5%. Calculate the value of each payment $x$, with Focal Points at Years 0, 1, 2, 3, 4. Build a table, compare, and give a general guess as to the pattern of changes in $x$.

5. Show that the ratio $R$ such that $R \left[ \frac{i}{1+(n-1)i} \right] = \left[ \frac{i}{1+ni} \right]$, is $R = \frac{1+(n-1)i}{1+ni}$.

6. Show that the difference $d$ between $i_n$ and $i_{n+1}$ where $i_{n-1} + d = i_n$ is $-i^2 \frac{1}{(1+ni)(1+(n-1)i)}$.

7. (a) Solve for $\Delta_n$, the growth rate of $1 at the simple interest rate $i$, where $(1 + \Delta_n) [1+(n-1)i] = 1+ni$. (b) Show that $1 + \Delta_n$ compounded from 1 to $n$ gives $1 + ni$. This is one connection between simple interest and compound interest.

8. Prove that for $S_c = P(1+r)^n$ and $S_s = P(1+ni)$, $S_c = S_s$ if and only if $i = \frac{(1+r)^n-1}{n}$ or $r = \sqrt[n]{1+ni} - 1$.

9. An investor opens an account with a deposit of $5100. Then deposits $3100 in one year, the $2100 in the next year for a balance of $12,000. What compound interest rate did the investor earn? (a) Draw a time line and pick a Focal Point which gives an Equation of Value with positive exponents. (b) Write and solve the equation. Use the TI84 Solver if you wish or solve by the quadratic formula. See the Side Bar Notes for code for the Solver.

10. Translate the problem in Exercise 9 to a problem about the times of the given yearly deposits one year apart to get 12.2%. Use the TI84 Solver to solve for the exponent $x$. 


Draw a time line with a Focal Point to interpret $x$ in terms of a certain year when a certain deposit occurs and then give the years of the other deposits in terms of $x$.

11. An Equation of Value with Unknown $n$: John wishes to save $4000 to start college at $100 at the end of each month in an account paying 6% per year compounded monthly. (a) How long will it take for him to reach his goal? Setup and solve an Equation of Value with Focal Point at the unknown number of months, $n$. (b) What is his balance when he reaches his goal? (c) Look up current savings rates and do the calculations. You may want to review you knowledge of the future value of an ordinary annuity. Solve with logarithms and with the TVM Solver. See the Side Bar Note on college education tax breaks.

12. A young man took out a sixty-month car loan with interest at 4% compounded monthly with monthly payments of $260.21. He missed the 13$^{th}$ and 14$^{th}$ payments. (a) Draw a time line with focal point at the end of the 36$^{th}$ month and 36$^{th}$ payment made. How much should he pay after the 36$^{th}$ payment is made to catch up on the loan payments? (c) What is his balance after catching up? (d) How much did he borrow for the car? (e) How much interest did he pay?
Answers:

1. (a) \[ 6000(1+.05)^6 + x = 4000(1+.05)^4 + 1000(1+.05)^3 + 5000(1+.05)^{-9}, \]
x = $1202.12.

(b) \[ 6000(1+6(.05)) + x = 4000(1+4(.05)) + 1000(1+3(.05)) + \frac{5000}{(1+9(.05))}. \]
x = $1598.28

3. (b) The effective rate from 0 to 1 = \( i_1 = i = .05 \). \( i_2 = \frac{i}{1+i} = .047619 \), \( i_3 = .0454545 \).
\( i_4 = .0434783 \). (c) \[ (1+.05)(1+.047619)(1+.0454545)(1+.0434783) = 1.20 = 1[1+4(.05)]. \]

4. Focal Point

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (1.05)(1.10) )</td>
<td>( (1.05)(1.05) )</td>
<td>1.10</td>
<td>1.15</td>
<td>1.20</td>
<td></td>
</tr>
<tr>
<td>2.15</td>
<td>2.05</td>
<td>2.05</td>
<td>2.15</td>
<td>2.25</td>
<td></td>
</tr>
<tr>
<td>$5372093</td>
<td>$5378049</td>
<td>$5365854</td>
<td>$5348837</td>
<td>$53333</td>
<td></td>
</tr>
</tbody>
</table>

7. (b) \[ (1+\Delta_1)(1+\Delta_2)(1+\Delta_3)...(1+\Delta_{n-1})(1+\Delta_n) = \]
\[ (1+i)^\left(\frac{1+2i}{1+i}\right)^{\left(1+3i\right)}\left(1+\frac{ni}{1+(n-1)i}\right) = 1+ni \]

9. \( i = 12.2\% \)

12. (b) To catch up after the 36th payment, he pays
\[ 260.21\left(1+\frac{.04}{12}\right)^{36-14} + 260.21\left(1+\frac{.04}{12}\right)^{36-13} \]
(c) The balance after catching up is
\[ 260.21\left[1 - \left(1+\frac{.04}{12}\right)^{-24}\right]. \]
References:


For a free course in financial mathematics with emphasis on personal finance, see [comap.com](http://comap.com). Click on the free financial mathematics course and register. COMAP will e-mail you a password. Simply click on an article in the annotated bibliography, download it, and teach it.

Unit 1: The Basics of Mathematics of Finance  Unit 2: Managing Your Money  Unit 3: Long-Term Financial Planning  Unit 4: Investing in Bonds and Stocks  Unit 5: Investing in Real Estate  Unit 6: Solving Financial Formulas for i.