The Lognormal Distribution (Preliminary Version)

Floyd Vest, June 2014

The lognormal distribution is applied in finance. It is commonly used on stock price per share, rate of return on stocks, earning earnings per share (EPS), Coefficient of Variation (CV), option pricing, and other applications, for which the distribution is skewed.

Definition of lognormal distribution. Given a parent normal distribution

(1) \[ f_N(y) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} \] we let

(2) \[ x = e^y \] so that \( y = \ln x \) and \( \ln x \) is normally distributed.

If a person knows calculus, they can derive the lognormal distribution of \( x \) to be

(3) \[ f(x) = \frac{1}{x\sigma \sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(\ln x - \mu)^2} \], \( x > 0 \). Both \( \mu \) and \( \sigma \) are from the parent normal distribution.

Useful formulas for the lognormal distribution can be derived. Using \( \mu \) and \( \sigma \) from the parent distribution, we have: (The following formulas have been copied into the article from Wikipedia.org, lognormal distribution, page 2.)

(5) \[ f(\text{Mode}) = \frac{1}{\sigma \sqrt{2\pi}} e^{\frac{\sigma^2 - \mu}{2}} \]

(6) Mean \( e^{\mu + \frac{\sigma^2}{2}} \)

(6.1) Median \( e^\mu \)

(4) Mode \( e^{\mu - \sigma^2} \)

(7) Variance \( (e^{\sigma^2} - 1)e^{2\mu + \sigma^2} \)

(9) Skewness \( (e^{\sigma^2} + 2)\sqrt{e^{\sigma^2} - 1} \)
A master moment generating function for the above moments is

\[ \mu_k = e^{\mu + \frac{1}{2}k^2\sigma^2}, k = 1,2,\ldots \]  For example, \( k = 1 \) gives

\[ \mu_1 = e^{\mu + \frac{1}{2}\sigma^2} \] which is the mean of the lognormal distribution.

Graphs on the TI 83/84.

Graphs of normal and lognormal distributions can be done on the TI 83/84. We will first work with the parent standard normal distribution with \( \mu = 0 \) and \( \sigma = 1 \). For the standard normal distribution where \( y \) is distributed normally

\[ f_N(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}, \text{ and the child lognormal distribution is} \]

\[ f(x) = \frac{1}{x\sqrt{2\pi}} e^{-\frac{(\ln x)^2}{2}}, x > 0. \]

To build a Window for the graph of this lognormal distribution it is helpful to know that Mode = \( e^{-1} = .3679 \) and from above \( f(\text{Mode}) = .6577 \).

Examination of \( f(x) \) will show that it is undefined at \( x = 0 \). We will take a look at

\[ f(.001) = \frac{1}{(.001)\sqrt{2\pi}} e^{-\frac{1}{2}(\ln .001)^2} = 1.735 \times 10^{-8}. \] This suggest that this graph starts close to (0,0).

With some experimentation, this gives for the two graphs the Window of \( x: -3, 5, 1 \) and \( y: 0, .7, .1 \).

You Try It #1

Using the above Formula 1 and 3 and Window, graph the parent standard normal distribution and the child lognormal distribution. Sketch and label the graphs on paper and put on units. Discuss the shapes, symmetry, mean, and maximum. You could use Trace for \( x = -4, \)
-3, 0, 3, 4, and 6, and calculate the values for the standard normal and lognormal. You will notice that as x gets larger, the lognormal approaches the x-axis slowly from above and that \( f(x) \) is positive.

You Try It #2

Enlarge and print from Wikipedia.org, page 4, the figure File: some log-normal distributions.svg. For the lognormal curve \( \mu = 0, \sigma = .25 \) graph both the parent normal distribution and the lognormal distribution on the same display. Label, sketch by hand, put on units, and discuss each and the difference. It interesting to see that the log normal is close to the normal and it doesn’t show on the graph close to (0,0). Evaluate the lognormal close to zero. For the lognormal, where is the Mean and Mode with respect to each other. Calculate the Skewness of the lognormal. Interpret.

You Try It #3

For the lognormal in You Try It #2, using the formula for Mode above, calculate \( f(\text{Mode}) \). Interpret.

You Try It #4.

Discuss why the lognormal distribution is not defined at \( x = 0 \), not defined for negative \( x \), and \( f(x) > 0 \).

You Try It #5.

Prove some of the properties of \( ln \ x \) which we may use in this article: \( \ln x^k = k (\ln x) \).

Define \( y = \ln x \). Calculate \( e \) and \( e^{-1} \). Graph \( e^x \). Graph \( e^{-x} \). Notice that replacing \( x \) with \( -x \) flips the curve 180 degrees about the y-axis.

You Try It #6.

Write the distribution formula for \( f(\ln x) \). Compare it to \( f(x) \). Discuss.

You Try It #7.

Use the moment generating function to generate the general moment formulas: Mean, \( k = 1 \); Variance, \( k = 2 \); Skewness, \( k = 4 \). Discuss the meaning of each.

**Application:**

From Lee, et. al., and Minitab we have price \( y \) for General Motors stock (GM), from 1975 to 1996 inclusive: Mean of \( y = 54.597 \) and Standard deviation of \( y = 15.916 \). For each \( y \) we calculate \( \ln y \) and \( \mu = \text{Mean}(\ln y) = 3.9575 \), and Variance \( (y) = \sigma^2 = .09256 \). Substitution into Formula 3 gives

\[
(15) \quad f(x) = \frac{1}{x(0.30423)\sqrt{2\pi}} e^{\left[-\frac{1}{2(0.0956)\ln(\ln x - 3.9575)^2}\right]} \text{ for the lognormal distribution for GM stock prices.}
\]

To graph, it is helpful to know that Mode = 47.7 and \( f(\text{Mode}) = .025 \). We could use
Window: x: 0, 90, 10 and y: 0, .03, .01. It might be helpful when putting \( f(x) \) in the calculator code, to calculate and put in the value of the complex numerical fraction.

Substitution into Formulas 6 and 7 above gives

\[
\mu_x = e^{\mu + \frac{\sigma^2}{2}} = 54.80 \text{ and } \sigma_x^2 = (e^{\sigma^2} - 1)(e^{2\mu + \sigma^2}) = 291.27, \sigma_x = 17.07. \quad (\text{Notice that we have a set of three mean and variance pairs.})
\]

You Try It #8.

Graph \( f(x) \) (Formula 15) above for GM stocks and its parent normal distribution. Sketch and label the two curves and discuss. Put on units. Calculate Skewness for \( f(x) \). The two curves are much alike. Give the differences and the various statistics. For the lognormal, what is the relationship between the Mean and Mode? Check \( f(x) \) for \( x \) values close to zero.

**Side Bar Notes:**

- **Requires calculus.** Derivation of the lognormal \( f(x) \) requires the Change of Variable Theorem from calculus.

- **Solvemymath.com.** See solvemymath.com, Main Calculator, Continuous Probability Distributions, Lognormal Distribution, and do the available calculations and report.

- **Better performing stocks.** Historical studies of the MSCI USA Quality Index of high Return on Equity (ROE) found that it outperformed over the last 25 years the S&P 500 by 1.7 percentage points. You can search for ROE stocks on YCharts.com or check holding by mutual funds which employ a high ROE strategy (CNNMoney.com, Aug. 2013, p 65. 66). For $100,000 over 25 years with a low rate of 9\% and a higher rate of (9+1.7)\%, what is the difference? What has been the Total return for the S&P 500, and the standard deviation as compared to the ROE Index? How do they compare in downside risk? See the article in this course “Annual Total Return Table for the S&P 500 Index of Stocks,” July 2013. There are many such historical studies on stocks. Has there been a three sigma event in the last 25 years? See investopedia.com for Return on Equity.

- **A changing of the balance in a portfolio.** From Jan.1, 2009 to March 31, 2014, the market changed a portfolio of 60\% stocks to 80\% stocks, and for bonds from 40\% to 27\%. What was the average annual increase for the value of the stock portfolio? For a balanced portfolio of one-third each of US stocks, Foreign stocks, and Bonds, US stocks increased to 44\%, Bonds declined to 22\%, what was the fractional change in Foreign stocks? (Money, June 2014, p. 50)

- **On a Money and happiness scale** for $13,200 per year income, happiness was 6+ and for $120,000 per year happiness was 8- (Money, June 2014).

- **Lognormal functions on Excel.** Do a Search for lognormal distribution on Excel. You will get information on lognormal functions on Excel. To watch and listen to videos on lognormal distributions in finance, go to mashpedia.com/Log-normal_distribution. Turn off the adds and you can choose from around twenty good videos based on Excel. If you take your time, you will understand the videos. He uses lognormal to predict future stock prices with confidence intervals.
Lognormal on TI86 and TI Nspire. We did a Search for lognormal distribution on TI 86 and one for TI Nspire and got no reference to lognormal. At deanza.edu, they had a list of probability distributions on the TI 86 and lognormal was not listed.

GM and Ford stocks. In this article we have data on prices of GM stocks from 1975 to 1996. More recently, from Dec. 2012 to April 2014, GM Price to Earnings ratio (P/E) has ranged from about 8.2 up to 8.5 and Ford from about 8.5 up to 11.3 (Money, June 2014, p. 56). You can get more precise numbers from finance.yahoo.com and Ibbotson.com and read about the success of each company. GM expects $1.3 Billion in recall costs but they have $20 Billion cash on hand. See the articles in this course on investing in stocks including “Valuation Formulas for Stocks.”

Buying a stock is buying a company. The task of a company is to make a profit for owners and shareholders by producing marketable goods and services. When you buy a stock you are buying a share of the company. Companies which offer stocks are required by the U. S. Securities and Exchange Commission (SEC) to file Financial statements: the Balance Sheet, Income Statement, and Cash flow Statement – annual and quarterly. When you get annual statements for a company which you own, these reports are likely to include hundreds of pages and hundreds of technical accounting terms. Many involve billions of dollars. The reports vary by the types of companies such as manufacturers or banks.

A good way to see a summary of financial statements is to go to the internet and put in Search YHOO: YAHOO INC, and you will see, before you click on the address, such multiples and ratios as Opening price, Volume of stock sales and trades, (P/E) Price to Earnings ratio, and Market Cap. You may have an idea about the meaning of these terms, but you might print out such terms found on some of the following pages for future study. You can look up the terms in investopedia.com and Wikipedia.org.

Click on the site address and you will see the front page giving many details for the Yahoo stock (ticker YHOO) and the company. Under Financial Statements, you can read summaries of Yahoo’s Income Statement, Balance Sheet, and Cash Flow. Even the term “Cash” has a complex meaning which includes cash, marketable securities, and other interest bearing securities. The basic formula for the Balance Sheet is: Assets = Liabilities + Equity (including stock equity). Take about thirty minutes or more to read as best you can the various entries and jot down or print out the terms you wish to look up.

Click on Stock Scouter Report and you will find a nice summary for the stock investor with a Quick Summary for Yahoo, Inc. You will appreciate the work of the stock analysts and mutual fund managers. Print this for future comparison. Then go back to Get Quote and put in T for AT&T and print out the Stock Scouter Report. Compare. Then since we have calculated lognormal distributions for GM stock prices, print out Stock Scouter Reports for GM and Ford, and compare. (Used textbooks on corporate accounting can be bought from barnesandnoble.com and amazon.com for less than the cost of shipping.)

You can get more information on history of stock prices, dividends, stock splits on the sites finance.yahoo.com and Ibbotson.com. See the articles in this course on stocks including “Valuation Formulas for Stocks.”
Exercises. Show your work. Label answers, numbers, and variables. When appropriate, write complete sentences. Name your calculation and graphing device. Explain some of your code.

#1. In general, what is Mean of \( \ln x \), Mean\[\ln x\]? What is Variance \[\ln x\]?

#2. Show that for the lognormal distribution, if Coefficient of Variation

\[
(CV) = \frac{\text{standard deviation}}{\text{mean}}, \text{ then } CV = \sqrt{e^{\sigma^2} - 1}.
\]

CV is unitless and allows the comparison of properties of values such as stock prices for different companies. High CV indicates high volatility.

#3. Write a formula for \( f(y) \) where \( y = \ln x \) and simplify.

#4. From Formulas 1 or 3, write a formula for \( f(ln x) \). What is the relationship between \( y \) and \( x \)? Give the mean and standard deviation.

#5. Use a calculator function to calculate the area under \( f(x) \) in Formula 13 from 0 to .3679. Shade this area on the graph.

#6. Calculate the area to the left of .69 for the normal distribution with \( \mu = 35 \) and \( \sigma = 2 \). Graph and shade.

#7. For a lognormal variable \( c \) whose parent is normal \( \mu = 0, \sigma = 1 \), calculate \( \mu_c \) and \( \sigma_c \).

#8. Graph the lognormal \( f(x) \) where \( \mu = 0 \) and \( \sigma^2 = .1 \).

#9. Graph the lognormal \( f(x) \) where \( \mu = 0 \) and \( \sigma^2 = 2 \).

#10. For lognormal show that \( \sigma^2 = \mu_c^2(e^{\sigma^2} - 1) \).

#11. Discuss the interpretations of Skewness coefficients.

#12. For a real world application, for the parent normal, \( \mu = 4.0749 \) and \( \sigma^2 = .08995 \). Calculate \( \mu_x \) and \( \sigma_x^2 \) for the lognormal distribution. Give the formula for \( f(x) \) and graph it. Discuss.

#13. Read the article in the References by Xiong about the TLF distribution and the lognormal distribution. Summarize. What distribution has thin tails, which has fatter tails? What do thin tails predict and what do fat tail predict? What are some of the historical downside events? What is meant by truncation? What is a three-sigma event? How does the TLF compare with the lognormal in representing historical downside risk? What are some long-term historical downside losses and their relative frequency? For the TLF distribution, see the article in this course “Investment Portfolio Design to Optimize Performance and Minimize Risk.” See investopedia.com for Black Swan Events and Investments. But, in eight out of ten geopolitical crises from 1940 to 2003, the S&P 500 more than recovered any losses by the six-month mark (Kiplinger’s Personal Finance, 7/2014, p. 30).
#14. Find the probability distribution for \( y = e^x \) where \( x \) is normally distributed with
Mean = \( \mu \) and Variance =\( \sigma^2 \).

#15. Income tax rates and intermediary drain rates: Scott says that for $100 per month invested over 40 years at 8 percent less expense ratio, for Variable annuities and small wrap accounts with an average expense ratio of 2.19%, the intermediary drain rate is 46%. This is higher than most peoples’ income tax rate. He uses the formula (Intermediary drain rate) =

\[ \frac{[(FV \text{ without expense ratio}) - (FV \text{ with expense ratio})]}{(FV \text{ without expense ratio})} \]

Does his formula make sense? Check his numbers. How much money is lost because of the expense ratio? What is the intermediary drain rate for an expense ratio of .1%, 1/10 th of one percent expense ratio? This expense ratio is not uncommon. (Scott Burns, Denton Record Chronicle, June 1, 2014, page 2D)

#16. (a) Make a probability distribution by putting together \( y = e^x \), \( x \leq 0 \), and \( y = e^{-x} \), \( x \geq 0 \). Choose a lower bound (lb) and upper bound (ub) so that the distribution is symmetrical about 0 and the total area under the curve is 1. Draw the graph and put on units. Notice that you have made a truncated distribution. (b) Slide the two curves down three units and make a probability distribution with a lb and an ub so that the area between the curves up to the x-axis is 1. (c) Flip (b) over 180 degrees about the x-axis. (d) Make your own probability distribution.

#17. A 13- Bagger and a 50-Bagger. A stock which cost $1.25 was sold two years later for $126.25, a 13-Bagger. In another deal, the investor from May 2009 to April 2014 made a 50-Bagger. What was the average annual rate of return on each?

#18. Berkshire Hathaway stocks had a real rate of return from 1976 to 2011 of 19%, the S&P 500 Index of stocks had a real rate of return of 6%. Go to USInflationcalculator.com and calculate the average annual inflation rate \( I \) for the period. Give rates of return without considering inflation. (Money.com, June 2014, p. 75) Stocks with tickers stating with W, A, R, R, E, N beat the market (Money.com, June 2014, p. 77), each group by at least 10 percentage points. S&P was 9%, 1994 to 2013. (This is called data mining.) See the article in this course “Annual Total Return for the S&P 500 Index of Stocks” and measuringworth.com to check their figures on the S&P 500.

#19. Use Mode = \( .3679 = e^{-1} \) in Formula 13 to calculate \( f(\text{Mode}) \). Why is this number helpful in building a Window for the graph of \( f(x) \)? Calculate Mean(x) and \( f(\text{Mean}) \). What are the relationships?

#20. Discuss how you would construct a confidence interval on a lognormal distribution. Describe the computer functions which would be useful. See the References.

References:

References in this course:
“Standard Deviation as a Measure of Risk for a Mutual Fund,” Summer, 2011.
“Covariance and Correlation in Finance,” June 2012.
“Confidence Intervals and Hypothesis Testing for Variance,” June 2012.

For a free course in financial mathematics, with emphasis on personal finance, for upper high school and college, see COMAP.com. Register and they will e-mail you a password. Simply click on an article in the annotated bibliography, download it, and teach it.

Unit 1: The Basics of Mathematics of Finance   Unit 2: Managing Your Money
Unit 3: Long-Term Financial Planning   Unit 4: Investing in Bonds and Stocks
Unit 5: Investing in Real Estate   Unit 6: Solving Financial Formulas for i. Unit 7: More Advanced or Technical. For about thirteen more advanced or technical articles see the UMAP Journal at COMAP.com.