The Mathematics of Refinancing

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Introduction

Financing describes a method of raising funds or capital. Many people finance an asset and pay for it in installments, as opposed to paying a sum up front. The cost of financing is the interest expense imposed by banks. This cost motivates people to search for the lowest interest rate possible. Even after a loan is taken, people look towards refinancing in order to achieve an even lower interest rate.

We model the effects of refinancing loans at lower interest rates. Mathematically, we show how the timing for refinancing that is most beneficial to the borrower depends on the terms left on the loan and the refinancing fees that could be imposed. Also, we look at how the refinanced loan is to be structured. We examine examples from the two most common financing types in the U.S., automobile and real estate loans. For instance, should the borrower always refinance if there is a lower interest rate? Can various fees make refinancing at lower interest rate worse than the original loan? This article will help to arrive at the answers to these questions.
Background

The Cost of Borrowing: Interest

Before discussing financing options, it is important to understand the cost of borrowing money, referred to as interest. If you invest money in a savings account, you earn interest, because the bank is paying you to borrow your money.

Interest can be calculated in two ways, as simple interest or compound interest. Simple interest is based only on the original principal amount while compound interest is applied to the principal plus the accumulated interest:

- For example, if you were to invest $1,000 into an account at 8% simple interest, your return would be $1,000 \times 0.08 = $80 per year for every upcoming year. Hence your balance after one year would be $1,080 and after one more $1,160 (= $1,000 + 2 \times $80).

- If the same amount were placed into an account at 8% interest compounded annually, the return would still be $80 in the first year. However, in the second year, the interest would be added to the current balance of $1,080 (the original $1,000 plus the interest of $80 added at the end of year one), hence the newly added interest would be $1,080 \times 0.08 = $86.40, making the balance $1,166.40—that is, $6.40 more than $1,160.

As time goes by, the balance gap between the two interest compoundings would grow larger, as Table 1 illustrates.

<table>
<thead>
<tr>
<th>Year</th>
<th>Simple Interest</th>
<th>Compound Interest</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
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<td>80</td>
</tr>
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</tr>
<tr>
<td>8</td>
<td>1,000</td>
<td>80</td>
</tr>
<tr>
<td>9</td>
<td>1,000</td>
<td>80</td>
</tr>
<tr>
<td>10</td>
<td>1,000</td>
<td>80</td>
</tr>
</tbody>
</table>
When an amount $S_0$ is invested (borrowed) at time 0 at the interest rate per term $i$, the return (debt) after $n$ terms is set up mathematically as follows.

- For simple interest:
  
  \[
  S(0) = S_0 \\
  S(1) = S_0 + iS_0 = S_0(1 + i) \\
  S(2) = S(1) + iS_0 = S_0(1 + i) + iS_0 = S_0(1 + 2i) \\
  S(3) = S(2) + iS_0 = S_0(1 + 2i) + iS_0 = S_0(1 + 3i) \\
  \vdots \\
  S(n) = S(n - 1) + iS_0 = S_0(1 + (n - 1)i) + iS_0 = S_0(1 + ni).
  \]

- For compound interest:
  
  \[
  S(0) = S_0 \\
  S(1) = S_0 + iS_0 = S_0(1 + i) \\
  S(2) = S(1) + iS(1) = S_0(1 + i) + iS_0(1 + i) = S_0(1 + i)^2 \\
  S(3) = S(2) + iS(2) = S_0(1 + i)^2 + iS_0(1 + i)^2 = S_0(1 + i)^3 \\
  \vdots \\
  S(n) = S(n - 1) + iS(n - 1) = S_0(1 + (n - 1)i) + iS_0(1 + i)^{n-1} = S_0(1 + i)^n.
  \]

Thus, the balance after $n$ terms for simple interest is

\[
S(n) = S_0(1 + ni)
\]

and for compound interest is

\[
S(n) = S_0(1 + i)^n.
\] (1)

**Exercise 1**

If $500 is deposited into a savings account with an annual interest rate of 6.5%, how much money including interest would you have in 5 years (assuming that you did not withdraw any money)? Consider both simple interest and compound interest and compare their end balances.

**Solution 1**

For simple interest: $S(5) = 500 \times (1 + 5 \times 0.065) = 662.5$.

For compound interest: $S(5) = 500 \times (1 + 0.065)^5 = 685.04$. 
When we plot the balance each year from Table 1, we get a collection of points. When we connect those points, we get a line for simple interest and a curve for compound interest (Figure 1). The line and curve represent the balance growth under simple interest and compound interest, respectively, with continuous time.

![Figure 1. Simple interest vs. compound interest.](image)

Throughout this article, we use compound interest and assume that the interest is compounded at the end of each term.

### How Lenders Make Money

When we intend borrow money, for example, to buy a house, we first would shop around and select lending agencies that offer low interest rate. We would calculate the monthly payment, then consider expenditures such as application fee and points (see below), as well as other conditions such as (possibly) a prepayment penalty. After comparing loans, we would choose the one with lowest cost.

On the other side of the cash flow, this cost becomes the lender’s revenue. Although Lender A offers may lower interest rate than Lender B, the loan application fee for Lender A may be higher than that for Lender B. Furthermore, Lender A may offer or request “points” that must be paid up front (at the closing of the loan), which may result in a higher total cost of the loan from Lender A despite the lower interest rate.\(^1\)

A *(discount)* point is 1% of the loan amount. In return for paying one point, we would receive a one-percentage-point reduction in the interest rate, which would reduce the monthly payment. For example, a 30-year fixed rate home mortgage for $100,000 at 8% would require a monthly payment of $733.76 (we explain later how to calculate this payment). However,

\(^1\)This is when the *annual percentage rate* (APR) comes in handy. It is the annual rate charged for borrowing, expressed as a single percentage number that represents the actual yearly cost of borrowing after any fees or additional costs; it gives borrowers the bottom-line cost of the loan.
a 30-year fixed rate mortgage for the same amount at 6% with 2 points would cost $599.55 per month, assuming that we pay the two points out of pocket (as opposed to having the cost of the points included in the balance of the loan). The difference in monthly payment is \( \$ (733.76 - 599.55) = \$ 134.21 \), so by paying \$100,000 \times 0.02 = \$2,000 \) up front, we save \$134.21 \) per month. In other words, we are paying part of the interest in advance so as to enjoy a lower monthly payment. After the recovery time of 15 months \((= \$2,000 \div \$134.21)\), we keep saving \$134.21 \) a month, so if we are to keep the loan longer than 15 months it is beneficial to pay two points at closing and make lower payments.

Another way that a lender makes money is through securitization, in which loans are bundled, the bundles are “sliced and diced,” and the results are repackaged into bonds or bundles of bonds, known as collateralized debt obligations (CDOs). Until recently, securitization of mortgages was extremely popular, because it benefited both lenders (usually banks) and borrowers:

- Bonds are easier to sell than mortgages, so banks could use their capital more efficiently by repackaging loans into bonds and selling the bonds to outside investors.

- Additionally, banks could collect fees for the repackaging transactions, further increasing their revenue.

- Most importantly, by selling bundled mortgages to outside investors, banks regained capital to extend more and cheaper loans to borrowers, including those who could not afford non-securitized mortgages.

Thus was sown the seed of the current mortgage crisis. Passing mortgage bundles to other investors meant also passing on the related risks of defaults on the loans, so the lending banks did not have to worry about the creditworthiness of borrowers—and hence went on lending sprees. On the other hand, borrowers believed that the real-estate bubble in the U.S. and many other parts of the world would last forever; so even though they were borrowing what they could not afford, they were counting on the value of their houses going up eventually, at which point they would be able to refinance. But the real-estate bubble burst in 2007, and the world has been going through the most severe economic downturn since the Great Depression.

Conceptually, securitization should result in efficient capital usage, yet in reality it was a vehicle to create too many loans that were almost guaranteed to default. An example is the adjustable-rate mortgage (ARM). An ARM is a loan in which the borrower makes monthly payments at a very low interest rate for a certain period of time (typically 1, 3, or 5 years), after which the loan changes to a fixed-interest loan at the prevailing interest rate at the reset time. A popular variation of the ARM is the interest-only ARM, in which the borrower pays only the accumulated interest and no principal
for an agreed number of months. Finally, the option ARM is one in which the borrower has an option of how much to pay each month: a traditional fixed-rate payment, only the interest, or a minimum payment that is even less. Typically, the minimum payment required is 1% to 2% of the loan total and is a “teaser rate.” The non-traditional loans allow the borrower to defer payment on the principal for an initial period of time. While the borrowers enjoy a lower payment for a short period of time, they eventually have to start paying the principal at the then-prevailing rate, which could make their monthly payment jump overnight. If the borrower cannot afford an increased payment or cannot refinance, the loan will default and the house will be foreclosed.

This sequence of events, repeated on a scale of millions of houses, is what caused the current financial crisis.

**Net Present Value**

The reverse of compounding interest is discounting, which is about establishing the present value of a future amount at annual interest rate $i$. If we want to receive $S_n$ at the end of the $n$th term, we should deposit an amount $S_0$ now such that $S_0(1 + i)^n = S_n$. This means that what is worth $S_n$ at $t = n$ is worth only $S_n/(1 + i)^n$ at $t = 0$. The value

$$
\frac{S_n}{(1 + i)^n}
$$

is called the present value (PV) of $S_n$.

**Net present value (NPV)** is a method that compares the value of a dollar today with the value of the same dollar at a future point in time. If someone were to give you a dollar today, it should be worth more than a dollar one month from now (if there is no inflation), because you could invest the dollar and gain interest over the month. More formally, NPV is the PV of future cash flows minus the present investment.

In business, NPV is an important tool in making investment decisions. For example, assume that an investment project has an expected yearly cash flow $(-10, 6, 3, 4)$, that is, you invest 10 at the beginning and collect 6, 3, and 4 at the end of years 1, 2, and 3, respectively. The NPV of this project is

$$-10 + \frac{6}{1 + i} + \frac{3}{(1 + i)^2} + \frac{4}{(1 + i)^3}$$

as opposed to the total $-10 + 6 + 3 + 4 = 3$ that does not take into account “the time value of money.” In general, when an investment project has an anticipated cash flow of $(c_0, c_1, c_2, \ldots, c_{n-1}, c_n)$, its NPV discounted at $i$ per term is

$$c_0 + \frac{c_1}{(1 + i)} + \frac{c_2}{(1 + i)^2} + \frac{c_3}{(1 + i)^3} + \cdots + \frac{c_{n-1}}{(1 + i)^{n-1}} + \frac{c_n}{(1 + i)^n}. \tag{2}$$
The sign of \( c_j \) can be anything: if positive, it is a return; if negative, it is an investment; if zero, it means that there is no financial activity during that term. Depending on the value of \( i \) and the \( c_j \), the NPV can be positive, negative, or zero. If the NPV of an investment is positive, you should invest; if it is negative, you should reject the investment; if it is zero, then you should consider other factors before making decision.

For example, let’s say that you own a coffee shop and are looking to buy another existing coffee shop in another town. First, you would want to estimate the future cash flow from the new shop; then you would want to discount that cash flow to find its present value. Let’s say that the present value is $600,000. If the other coffee-shop owner would sell the shop to you for less than $600,000, it would be worth buying because the NPV (cost of shop minus present value of future cash flow) is positive. However, if the owner wanted more than $600,000, the purchase would not be a good investment because the NPV is negative, meaning you would be paying more money for something than its future returns are worth now.

**Exercise 2 Net Present Value**

a) If someone were to give you $1,000 two years from now, what would it be worth today, given that you could invest the money at an interest rate of 4.2%?

b) You have an opportunity to invest $400. The expected yearly cash flows are \((-400, 100, 150, 200)\). Is the investment worth it, given an interest rate of 5.5%?

**Solution 2**

a) Let \( M \) be the worth today of the $1,000 from the future. Start by finding the total interest earned. Recall that we use compound interest; therefore, the interest gained in year 2 equals the interest gained on principal and the interest gained on the interest from year 1.

\[
\text{Total interest} = \text{Interest Gained in Year 1} + \text{Interest Gained in Year 2} = (M \times 0.042) + (M \times 0.042) + 0.042(M \times 0.042) = 0.084M + 0.001764M.
\]

To find the worth of 1,000 today, you take \( M \) plus all the interest on \( M \) and set it equal to 1,000:

\[
M + 0.084M + 0.001764M = 1,000,
\]

which gives us \( 1.085764M = 1,000 \), and by dividing both sides by 1.085764 we get

\[
M = 921.01.
\]
This is indeed (1) (on p. 431) in another form. Set

\[ M(1 + 0.042)^2 = 81,000 \]

and solve for \( M \); you end up with the same number.

b) By (2) (on p. 434), we have

\[
\text{NPV} = -400 + \frac{100}{1 + 0.055} + \frac{150}{(1 + 0.055)^2} + \frac{200}{(1 + 0.055)^3}
\]

\[
= -400 + 94.79 + 134.77 + 170.32
\]

\[ = -0.12. \]

Because the NPV is (barely) negative, it is not worth making the investment.

**Conventional Loans**

**Real-Life Questions**

Most loans used to purchase a home, buy a car, or pay tuition are “conventional” loans. This means that they are fully-amortized loans: They have periodic payments of equal size, each of which pays the interest accumulated during the period (usually one month) and part of the remaining principal.

We are interested in applying our theory to real-life problems, so we assume that each period is one month long.

The payment \( A \) per term is determined by the principal \( P \), the number \( N \) of terms, and the interest rate \( i \) per term.

To determine the payment \( A \) from the conditions of the loan, consider the following two financial activities.

- First, open an account: Deposit \( P \) at time zero and let it grow at the interest rate \( i \) per term.
- At the same time, open another account and keep depositing \( A \) at the end of each term (Figure 2).

After \( N \) periods, compare the balances of the two accounts.

- Using compound interest, the balance of the first account at the end of the \( N \)th period is

\[ P(1 + i)^N. \]
For the balance of the second account, we apply the same concept for each deposit:

1. The first deposit of $A$ will have grown to $A(1 + i)^{N-1}$;
2. the second deposit of $A$ will have grown to $A(1 + i)^{N-2}$, since the second $A$ has been in the account for one fewer period than the first $A$;
3. similarly, the third deposit of $A$ will have grown to $A(1 + i)^{N-3}$.

Continuing in this manner, the second-from-the-last deposit will have just one period for compounding, and the last one no period at all. Therefore, the balance of the second account at the end of the $N$th period is

$$A(1 + i)^{N-1} + A(1 + i)^{N-2} + \cdots + A(1 + i) + A = A \frac{(1 + i)^N - 1}{i}.$$

Now apply the tales of these two accounts to solve our problem. Once we finish depositing $N$ payments, we use the lump sum of

$$A \frac{(1 + i)^N - 1}{i}$$

to pay off $P(1 + i)^N$. Thus, we solve

$$P(1 + i)^N = A \frac{(1 + i)^N - 1}{i}$$

for $A$ to find the monthly payment, hence

$$A = P \frac{(1 + i)^N}{(1 + i)^N - 1} = P \frac{i}{1 - (1 + i)^{-N}}.$$

This kind of cash flow—a series of payments of equal size—is called an *annuity*. Most loan payment schedules and pension plans (in which you deposit a certain amount each month to take out a lump sum at retirement) are examples of annuities.

Now, with our newly-gained knowledge, let us see whether we can answer the following questions.
Assume that we need to finance $100,000 for 240 months at 0.3% per month and we pay it off with an annuity. Assume that we do not have any cash at hand, so if there is any financing fee\(^2\) that we have to pay up front, we should borrow it too and spread its payment throughout the loan—which is equivalent to just borrowing more to begin with. In real life—for example, in buying a house—the fees are supposed to be paid up front, at the time of closing, as part of the closing costs, instead of becoming part of the principal; sometimes people get a second loan to finance the closing costs.

We simplify the situation and assume that there is only one loan and one bank that we borrow from. We also assume that there is no inflation.\(^3\)

**Question 1**

When no financing fees (e.g. application fee, points) are involved, what are the monthly payment and the total amount paid throughout the loan period?

**Question 2**

When there is an application fee of $2,000, what is the monthly payment? How much more money are we paying in the long run than with no fees?

**Question 3**

When there is a 3-points (recall that one point is 1% of the loan amount) financing fee, what is the monthly payment? How much more money are we paying in the long run than with no points?

**Question 4**

When there are an application fee of $2,000 and a 3-points financing fee, what is the monthly payment? How much more money are we paying in the long run than with no fees and points?

The only differences among these four questions is how much we need to borrow at the beginning. Once we know the loan amount, we can apply (3) (on p. 437) to get our monthly payment.

The answers begin on the following page.

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\(^2\)A financing fee is the money paid (usually up front) to get financing, while the similar-sounding finance charge is anything added to the principal, such as interest, application fee, service fee, financing fee, late fee, etc.

\(^3\)When there is inflation measured by the yearly rate \(r\), the inflation-adjusted (or real) interest rate \(j\) and the interest rate \(i\) are related by \(1 + j = (1 + i)/(1 + r)\).
Answer 1

In 240 months, your debt will have grown to $100,000(1 + 0.003)^{240}$. If we pay \( A \) per month at the end of each month for 240 months, the future value of this annuity is

\[
A \left( \frac{(1 + 0.003)^{240} - 1}{0.003} \right).
\]

Solving

\[
$100,000 (1 + 0.003)^{240} = A \left( \frac{(1 + 0.003)^{240} - 1}{0.003} \right)
\]

for \( A \), we find that the monthly payment is \( A = 585.11 \). The total of all payments throughout the loan is $585.11 \times 240 = $140,426.40 (see Figure 3).

<table>
<thead>
<tr>
<th>Time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...............</th>
<th>239</th>
<th>240</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>100,000</td>
<td>-585.11</td>
<td>-585.11</td>
<td>...............</td>
<td>-585.11</td>
<td>-585.11</td>
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</table>

Figure 3. Amortization without any initial fee.

Answer 2

Now the initial borrowing is $(100,000 + 2,000) = $102,000. So we solve

\[
$102,000(1 + 0.003)^{240} = A \left( \frac{(1 + 0.003)^{240} - 1}{0.003} \right)
\]

for \( A \) to get the monthly payment \( A = 596.81 \). The total extra cost over the loan term due to the fee is the total payment less what would be the total payment without the fee, hence

\[
\text{Extra payment} = \text{total with fee} - \text{total with no fees}
\]

\[
= 596.81 \times 240 - 140,426.40
\]

\[
= 2,808.
\]

See Figure 4.

Answer 3

It is tempting to think of 3 points being 3% of the principal, so that the fee is \( 0.03 \times $100,000 = $3,000 \) and the initial borrowing is

\[
$(100,000 + 3,000) = $103,000.\]
However if we pay 3% of this amount, we are left with

\[ $(103,000 - 3,090) = $99,910, \]

still short $90! This is because we have to pay 3% as well on the extra money borrowed to be paid as points. So instead of just adding $3,000, we have to set the amount to be borrowed as \( x \) and solve \( x - 0.03x = $100,000 \) for \( x \). We get \( x = $100,000 ÷ 0.97 = $103,093 \). The monthly payment $603.21 is obtained by solving

\[
$103,093(1 + 0.003)^{240} = A \frac{(1 + 0.003)^{240} - 1}{0.003}
\]

for \( A \). The extra payment due to the points is the total payment less what would be the total payment without the points, hence

\[
\text{Extra payment} = \text{total with points} - \text{total with no points} \\
= $603.21 \times 240 - $140,426.40 \\
= $4,344.
\]

See Figure 5.

```
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<thead>
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<th>2</th>
<th>239</th>
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Figure 4. Amortization with a $2,000 fee.

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<th>1</th>
<th>2</th>
<th>239</th>
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Figure 5. Amortization with 3 points.

**Answer 4**

This time we need to borrow enough money so that after paying 3 points we still have $100,000 plus the fee $2,000, therefore we must solve \( x - 0.03x = $102,000 \) for \( x \). We get \( x = $102,000 ÷ 0.97 = $105,155 \). The monthly payment is $615.27, from solving

\[
$105,155(1 + 0.003)^{240} = A \frac{(1 + 0.003)^{240} - 1}{0.003}
\]
for $A$, and the extra payment is

\[
\text{Total with fees and points} - \text{total with no fees or points} = 615.27 \times 240 - $140,426.40 = $7,238.40.
\]

See Figure 6.

\[
\begin{array}{cccccc}
\text{Time} & 0 & 1 & 2 & \ldots \ldots & 239 & 240 \\
\hline
105,155 & -615.27 & -615.27 & \ldots \ldots & -615.27 & -615.27 \\
\end{array}
\]

**Figure 6.** Amortization with $2,000$ initial fee and 3 points.

The questions above all deal with the same interest rate. How about loans with different interest rates and financing fees? When is one loan better than another? Consider the following question.

**Question 5**

Suppose that we need to borrow $200,000 for 360 months and we have a choice: 7.2\% per year (so 0.6\% per month), or 6\% per year (so 0.5\% per month) with 2 points. Which should we choose?

**Answer 5**

For the 7.2\% loan, the monthly payment is $1,357.58 by solving for $A$ in

\[
$200,000(1 + 0.006)^{360} = A \frac{(1 + 0.006)^{360} - 1}{0.006}.
\]

For the 6\% loan, the loan amount is $204,082. The monthly payment is $1,223.57 by solving for $B$ in

\[
204,082(1 + 0.005)^{360} = B \frac{(1 + 0.005)^{360} - 1}{0.005}.
\]

The total cost for the 7.2\% loan is

\[
$1,357.58 \times 360 = $488,728.80,
\]

and the total cost for the 6\% loan is

\[
$1,223.57 \times 360 + 0.02 \times $200,000 = $444,485.20.
\]

Therefore, by choosing the 6\% loan with 2 points, we can save

\[
$(488,728.80 - 444,485.20) = $44,243.60.
\]
Effective Rate of Interest

Now that we have studied five real-life questions, we may ask the following question: Why do lenders charge financing fees, and exactly how do they work?

From the examples, we see that not only the interest rate but also the financing fees should be considered when taking out a loan. In real life, there are a lot of banks, and each offers many financing products; it is nearly impossible to compare the details of all of them. Therefore, we need a standard measure that can be used for all financing products regardless of their payment and fee schedule. It is called the effective rate of interest, and it is the interest rate that makes the NPV of the cash flow of the loan zero. More precisely, assume that we have a loan package with a cash flow of $(c_0, c_1, c_2, \ldots, c_{n-1}, c_n)$ (see Figures 3–6). Then the effective rate of interest $i_e$ satisfies

$$0 = c_0 + \frac{c_1}{1 + i_e} + \frac{c_2}{(1 + i_e)^2} + \cdots + \frac{c_{n-1}}{(1 + i_e)^{n-1}} + \frac{c_n}{(1 + i_e)^n}.$$

For simple cash flows, we can find $i_e$ with a financial calculator or with the built-in functions of a spreadsheet. However, for complicated cash flows that go beyond the range of such devices, we need to use a more sophisticated method, such as computer programming.

Is Every Refinancing Good?

Over the past several years, many people refinanced loans, taking advantage of historically-low interest rates. But many of those people are in trouble, even facing foreclosure, victims of adjustable-rate mortgages. After the troublesome 2007 and 2008, interest rates are low again in the fourth quarter of 2009, and refinancing applications have risen. But is it always a good idea to refinance as long as the interest rate gets lower? Consider the following examples, both of which are very common real-life financing situations.

These examples contain new vocabulary that is standard in banking. A nominal rate is the interested rate quoted by the bank regardless of the compounding frequency; per annum is Latin for “per year.”

- Automobile loan: Consider an auto loan in which we borrow $20,000 for five years at a nominal rate of 9% per annum with no financing fee. If we have an opportunity to refinance the loan for the rest of the term of the loan (that is, staying with the original total number of periods) at 4.5% per annum with a fee, should we refinance or stick with the old loan? What about for other interest rates?

- Home mortgage: Consider a 30-year home mortgage loan with $300,000 principal at 7.8% per annum. If we have an opportunity to refinance at a
lower interest rate with an application fee of $500 and 3 discount points (which is very common in real estate refinancing), should we always refinance?

We have no idea how to answer. In fact, there are no definite yes-or-no answers to the questions as they are, for the answers depend on many factors, such as the new interest rate, the duration of the original loan, the number of remaining terms, the length of the new loan, or all of these. Rather than working on problems case by case, we will build refinancing models that will answer our questions all at once.

Mathematical Setup

Present Value Interest Factor

As we have seen, the PV and NPV play important roles in finance. To calculate the PV of a cash flow \((c_0, c_1, c_2, \ldots, c_{N-1}, c_N)\), we need the geometric series

\[
\frac{1}{1 + i} + \frac{1}{(1 + i)^2} + \cdots + \frac{1}{(1 + i)^N} = \frac{1}{i} \left\{1 - \frac{1}{(1 + i)^N}\right\}.
\]

This is the present value of an annuity of $1 for \(N\) months discounted at \(i\) per month, and in finance it is called the present value interest factor for an ordinary annuity. In actuarial science, this amount is denoted by the symbol \(a_{N|i}\) (read “\(a\ angle N\ at i\)”). For a fixed \(i\), the more terms there are, the larger \(a_{N|i}\) becomes. On the other hand, for a fixed \(k\), a bigger \(i\) means a smaller \(1/(1 + i)^k\). Therefore, \(a_{N|i}\) is increasing in \(N\) and decreasing in \(i\).

Amortization with Financing Fees

We assume the following as a basis for all the cases that we consider.

- Principal: \(P\)
- Nominal interest rate per month: \(i\)
- Life of the loan in months: \(N\)
- Monthly payment, made at the end of each month: \(A\)

The structure of the flow of payments is shown in Figure 7.
The payment $A$ is determined such that the sum of the present values of the payments equals the principal (compare with (3) on p. 437),

$$P = A \frac{1 - (1 + i)^{-N}}{i} = A a_{N|i}. $$

In most real-life situations, the loan structure is more complicated. Usually, there are up-front financing fees and discount points so that the lending agencies can make more profit. When we want to have $P$ at our disposal, we need to borrow more to cover all the fees; therefore, the new principal $Q$ becomes greater than $P$ and we make a monthly payment of

$$A = \frac{Q}{a_{N|i}} > \frac{P}{a_{N|i}}. $$

The higher payment for the same net amount borrowed raises the actual interest rate applied to the borrower, which is called the effective rate of interest. Recall that this is the interest rate that makes the NPV to the borrower (hence also to the lender) zero. When there is no fee, $Q = P$ and by definition the effective rate of interest is the same as the nominal rate $i$.

### Amortization with an Application Fee $F$

- New principal: $Q = P + F$
- New payment: $A_f = \frac{P + F}{a_{N|i}}$
- Effective cost of debt $i_f$ is determined by

$$0 = P - A_f \left\{ \frac{1}{1 + i_f} + \frac{1}{(1 + i_f)^2} + \cdots + \frac{1}{(1 + i_f)^n} \right\}. $$

### Amortization with $x$ Discount Points

One discount point is 1% of the principal, so we can treat points just like percentage points and write $x$ points $= x/100$. When a principal $Q$ is loaned with $x$ points, the borrower is supposed to pay an up-front fee of
Figure 8. Full Amortization with an up-front fee $F$.

$xQ$ to the lending agency, hence $Q(1-x)$ is the net amount available to the borrower; thus, $P = Q(1-x)$ and we have the following:

- New principal: $Q = \frac{P}{1-x}$
- New monthly payment: $A_p = \frac{Q}{a_N i}$
- Effective cost of debt $i_p$ is decided by

$$0 = P - A_p \left\{ \frac{1}{1 + i_p} + \frac{1}{(1 + i_p)^2} + \cdots + \frac{1}{(1 + i_p)^N} \right\}.$$  

Figure 9. Full amortization with $x$ discount points.

Amortization with Combined Fees: $F$ plus $x$ Discount Points

When $x$ discount points and a financing fee $F$ are imposed on $Q$ at $t = 0$, the borrower is supposed to pay both $xQ$ and $F$ up front, making $Q - xQ - F = P$. Therefore, we have the following:

- New principal: $Q = \frac{P + F}{1-x}$
- New monthly payment: $A_c = \frac{Q}{a_N i}$
- Effective cost of debt $i_c$ is determined by

$$0 = P - A_c \left\{ \frac{1}{1 + i_c} + \frac{1}{(1 + i_c)^2} + \cdots + \frac{1}{(1 + i_c)^N} \right\}.$$  

(4)
Refinancing

Now we apply the concepts from the previous section to our main topic. Consider a loan such that

- Principal: \( P \)
- Nominal rate: \( i \) per month
- Loan period: \( N \) months
- Monthly payment: \( A = \frac{P}{a_{N|i}} \) paid at the end of each month

Even when there are a financing fee \( F \) and \( x \) points, we can replace \( i \) by \( i_c \) and \( A \) by \( A_c \) in (4) to make it a zero-fee loan. With this reconceptualization, we can assume no financing fee at the beginning.

With each monthly payment, the borrower pays back part of the principal. The total part of the principal that has been paid is called equity, which is the principal less the outstanding debt. After making \( N - n \) payments, there still are \( n \) more payments left, thus the remaining debt is the present value of the annuity of \( A \) for \( n \) terms, so

\[
\text{Outstanding balance} = A \frac{1 - (1 + i)^{-n}}{i} = A a_{n|i}. \tag{5}
\]

The equity built during the first \( N - n \) payments in the above loan is the principal less the remaining debt, hence

\[
\text{Equity} = P - A \frac{1 - (1 + i)^{-n}}{i} = P - A a_{n|i}.
\]

When refinancing, there are two cases.

- **Continue the same loan at a lower interest rate**, with a new lower payment while paying off the debt during the remaining term. This works like a payment reset, and as a result we finish paying off the loan as scheduled. This kind of refinancing is common for small loans with a short loan term, such as automobile financing. Many auto dealers have
their own financing programs and often offer a lower interest rate during the life of a loan in return for a small refinancing fee. We can apply the same idea to a home owner who refinances a 30-year home mortgage after 15 years and continues with a 15-year mortgage.

- **Start a new loan to pay off the old loan.** Most real-estate refincancings are done this way. However, not everybody uses all of the equity for refinancing. A lot of people borrow as much as possible, using their house as collateral, to have extra cash for other things such as purchases, medical expenses, and paying off credit-card debt. We assume refinancing only for the house, therefore using all the equity on it for refinancing. Also, we assume that the new loan has the same length as the original one, so that the total length of the combined loan increases.

To build mathematical models of the two types of refinancing, we assume the following.

**Assumptions for Refinancing**
- Number of payments made by the time of refinancing: \( N - n \)
- Interest rate per month for refinancing: \( i_{\text{new}} \), which is lower than \( i \)
- All the equity built to date is used for refinancing.
- Refinancing fee: \( F \)
- Discount points for refinancing: \( x \)

As we deal with several loans with various fees, terms, and interest rates, we are inundated with notation. To help avoid confusion, here is the rationale behind the nomenclature.

**Remark on Notations**
- \( P \) denotes the original principal before refinancing (\( P \) for principal).
- \( Q \) denotes the new principal for refinancing (alphabetical order: \( Q \) comes after \( P \)).
- \( A \) denotes the original monthly payment before refinancing (\( A \) for amount).
- \( B \) and \( C \) denote the new respective monthly payment after refinancing (alphabetical again: \( B \) comes after \( A \) and \( C \) after \( B \)).
- \( D \) denotes the outstanding debt at the time of refinancing (\( D \) for debt).
- When the above items depend on the number \( n \) of remaining periods for the original loan at the time of refinancing, they are outfitted with a subscript \( n \), such as \( B_n, C_n, D_n \), etc.
Picking up the Old Loan: Short-Term Refinancing

We are “picking up” the old loan, so the number of remaining payments is \( n \). By (5) and the results from *Amortization with Combined Fees* on p. 445, we have the following results:

- Remaining debt: \( D_n = A \frac{1 - (1 + i)^{-n}}{i} = A a_{\bar{m}_i} \)
- New loan principal: \( Q_n = \frac{D_n + F}{1 - x} \)
- New loan period: \( n \) months
- New monthly payment: \( B_n = \frac{Q_n}{a_{\bar{m}_{\text{new}}}^n} \)

Once the monthly payment is fixed, we can express the net present value to the borrower as a function of the nominal interest rate \( i \):

\[
\text{NPV}_s(i) = D_n - B_n \left\{ \frac{1}{1 + i} + \frac{1}{(1 + i)^2} + \cdots + \frac{1}{(1 + i)^n} \right\} = D_n - B_n a_{\bar{m}_i}.
\]

Let \( i_s \) be the effective rate of interest of this short-term refinancing (the index \( s \) is for “short”). By definition of the effective rate of interest \( i_s \),

\[
\text{NPV}_s(i_s) = D_n - B_n a_{\bar{m}_i} = 0,
\]

\[
\text{NPV}_s(i_{\text{new}}) = D_n - B_n a_{\bar{m}_{\text{new}}} = D_n - Q_n = D_n - \frac{D_n + F}{1 - x} < 0.
\]

The refinancing is logical only when \( i_s < i \), or equivalently,

\[
\text{NPV}_s(i) > 0 = \text{NPV}_s(i_s),
\]

because \( \text{NPV}_s(i) \) is an increasing function of \( i \). (Why? Think about \( a_{\bar{m}_i} \).)

We calculate to determine when that condition holds:

\[
\text{NPV}_s(i) = D_n - B_n a_{\bar{m}_i} = A a_{\bar{m}_i} - \frac{Q_n}{a_{\bar{m}_{\text{new}}}^n} a_{\bar{m}_i} = \left( A - \frac{Q_n}{a_{\bar{m}_{\text{new}}}^n} \right) a_{\bar{m}_i} > 0,
\]

which holds if

\[
A - \frac{Q_n}{a_{\bar{m}_{\text{new}}}^n} = A - \frac{A a_{\bar{m}_i} + F}{1 - x} \cdot \frac{1}{a_{\bar{m}_{\text{new}}}^n} > 0.
\]

This is equivalent to

\[
A[(1 - x) a_{\bar{m}_{\text{new}}} - a_{\bar{m}_i}] > F.
\]
Since $a_{n|i}$ is a decreasing function of $i$, $i_{\text{new}} < i$ implies $a_{n|i_{\text{new}}} > a_{n|i}$, and the size of left hand side of (6) is decided by the number $x$ of discount points. Thus, it is worth refinancing only when

$$F < A[(1 - x)a_{n|i_{\text{new}}} - a_{n|i}].$$

### Starting a New Loan: Long-Term Refinancing

We use the equity from the previous loan to start a new loan for another $N$ months, so the only thing that changes for this long-term refinancing is the number of remaining payments.

- Remaining debt: $D_n = A \frac{1 - (1 + i)^{-n}}{i} = A a_{n|i}
- New loan principal: $Q_n = \frac{D_n + F}{1 - x}$
- New loan period: $N$ months
- New monthly payment: $C_n = \frac{Q_n}{a_{N|i_{\text{new}}}}$

With these notations, the net present value for the borrower is

$$\text{NPV}_{l}(i) = D_n - C_n \left\{ \frac{1}{1+i} + \frac{1}{(1+i)^2} + \cdots + \frac{1}{(1+i)^N} \right\} = D_n - C_n a_{N|i}.$$

Let $i_l$ be the effective rate of interest of this long-term refinancing (the index $l$ is for “long”). By definition of $i_l$, $\text{NPV}_{l}(i_l) = D_n - C_n a_{N|i_l} = 0$.

As before, $i_l < i$ if and only if $\text{NPV}_{l}(i) > 0$. We have

$$\text{NPV}_{l}(i) = D_n - C_n a_{N|i} = A a_{n|i} - C_n a_{N|i} = A a_{n|i} - \frac{Q_n}{a_{N|i_{\text{new}}}} a_{N|i} = A a_{n|i} - \frac{D_n + F}{(1 - x)a_{N|i_{\text{new}}}} a_{N|i} = A a_{n|i} - \frac{A a_{n|i} + F}{1 - x} \frac{1}{a_{N|i_{\text{new}}}} a_{N|i} > 0$$

if and only if

$$A a_{n|i} \left\{ (1 - x)a_{N|i_{\text{new}}} - a_{N|i} \right\} > F a_{N|i}.$$
or

\[ F < \frac{A \alpha_i}{\alpha_i} \left\{ (1 - x) a_{N i_{\text{new}}} - a_{\alpha_i} \right\}. \]  

(8)

When \( n = N \), the conditions (7) and (8) become identical, that is,

\[ F < A \left\{ (1 - x) a_{\alpha_i_{\text{new}}} - a_{\alpha_i} \right\}. \]  

(9)

This is not a refinancing condition but rather a condition for another loan with an up-front fee \( F \), discount points \( x \), and an interest rate \( i_{\text{new}} < i \) to be more profitable than continuing the original loan.

**To Refinance or Not**

Now that we know the structure of refinancing, we have a better judgment of whether to refinance or not and are ready to answer the questions on pp. 442–443 about whether it is a good idea to refinance. To reflect the real-life situations, we will use short-term financing for the automobile loan and long-term one for the house loan.

**Short-Term Refinancing: An Automobile Loan**

Consider the automobile loan from p. 442 with $20,000 borrowed for five years at a nominal rate of 9% per annum (hence \( r = 0.75\% \)). When there is no financing fee, the monthly payment \( A \) is

\[ A = \frac{P}{\alpha_i} = \frac{P_i}{1 - (1 + i)^{-N}} = \frac{20,000 \times 0.0075}{1 - (1 + 0.0075)^{-60}} = \$415.17. \]

If the borrower has an opportunity to refinance the loan for the rest of the loan period (that is, picking up the original period) at 4.5% per annum (\( i_{\text{new}} = 0.375\% \)) with a fee, should the borrower refinance or stick with the old loan?

The answer depends on the number of remaining months \( n \) and the refinancing fee \( F \). With the monthly payment \( A \), we use the formulas in **Picking Up the Old Loan** on p. 448 to get the components in **Table 2**, where we set \( F = \$250 \).

Additionally, we can calculate the cumulated interest by subtracting the principal (and financing fees, if there is any) from the total payment. Under the original loan, the cumulated interest would be simply \( NA - P \). However, in the case of refinancing, it becomes

\[ (N - n) A + n B_n - (P + F). \]  

(10)
We saw in (7) that as long as
\[ A_N \left\{ \left(1 - x\right) a_{m_{\text{new}}} - a_{m_i} \right\} > F = $250, \]
the effective rate of interest per month \( i_s \) is less than the nominal rate \( r = 9\%/12 = 0.75\% \). For example, referring to the table, if the borrower refinances after 3 years (with 24 payments left), the new effective rate of interest is 7.17\% per annum and the new payment is $407.57 for the rest of the loan period (that is, the remaining 24 payments). Note that even at a new rate, just half of the original rate, the monthly payment is not as low as one may think: The difference in payments is $(415.17 - 407.57) = $7.60, not even $10! If the borrower refinances one year earlier, after 2 years (with 36 payments left), then the new effective rate of interest is 5.77\%, the new payment is $395.80, and the difference in payments is $(415.17 - 395.80) = $19.37, a little bit less than $20.

Why is this happening? It is because the loan is a fully-amortized loan. At the beginning of a fully-amortized loan, most of the monthly payment goes towards paying the interest on the unpaid part of the principal. As more and more payments are made, the equity (the total paid part of the principal) grows, so the interest portion of the monthly payment decreases. If the borrower refinances early in the term of the loan, the equity built is still small, and the new principal plus the refinancing fee is not as small as one might expect. Still, it is worth refinancing as long as the effective rate of interest after refinancing is less than the original nominal rate, because then the total cumulated interest after refinancing is less than that under the original loan. We should be careful; the cumulated interest when refinancing after 4 years is $4,798, which seems lower than that under the original loan, $4,910; therefore, refinancing regardless of the effective rate of interest still looks like a good deal. But the amount $4,798 does not include the
financial fee of $250; with this fee added in, the total cost of the borrowing is $(4,798 + 250) = 5,048 > 4,910.

Table 3 shows the case with a higher refinancing fee of $F = 500. Both the effective rate of interest $i_{e.s}$ and the new monthly payment $B_n$ are greater than for the case $F = 250.$

<table>
<thead>
<tr>
<th>Years paid</th>
<th>Payment $A$</th>
<th>1</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payments remaining $n$</td>
<td>48</td>
<td>36</td>
<td>30</td>
<td>24</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>Debt remaining $D_n$</td>
<td>16,683</td>
<td>13,056</td>
<td>11,116</td>
<td>9,088</td>
<td>4,747</td>
<td></td>
</tr>
<tr>
<td>New principal $Q_n$</td>
<td>17,183</td>
<td>13,556</td>
<td>11,616</td>
<td>9,588</td>
<td>5,247</td>
<td></td>
</tr>
<tr>
<td>New payment $B_n$</td>
<td>391.84</td>
<td>403.24</td>
<td>410.12</td>
<td>418.48</td>
<td>448.02</td>
<td></td>
</tr>
<tr>
<td>Effective rate/yr $12 \cdot i_s$</td>
<td>6.00%</td>
<td>7.02%</td>
<td>8.01%</td>
<td>9.79%</td>
<td>23.61%</td>
<td></td>
</tr>
<tr>
<td>Upper bound of $F$ in (7)</td>
<td>1522.90</td>
<td>900.96</td>
<td>643.02</td>
<td>424.10</td>
<td>115.26</td>
<td></td>
</tr>
</tbody>
</table>

We can verify that when the fee $500 is greater than the upper bound in (7) (refinancing after 3 years and 4 years, respectively), the effective rate of interest $i_s$ is greater than the nominal rate $i = 9\%$, so it is not worth refinancing. Again, we should not forget that the cumulated interest after refinancing does not include the refinancing fee of $500; so for refinancing after 3 years, the cost of borrowing is $(4,490 + 500) = 4,990$, and after 4 years, $(4,804 + 500) = 5,304$, both of which are greater than $4,910$, the original interest amount.

**Long-Term Refinancing: Starting a New Loan**

The main difference between short-term refinancing and long-term refinancing is the length of the refinanced loan. While a short-term refinancing picks up the original loan period, long-term refinancing starts a new loan. As a result, the monthly payment will be reduced, even at a slightly lower interest rate, as long as no more borrowing is required. On the other hand, extended loan periods may result in greater cumulated interest. For example, consider a refinancing of the home mortgage loan introduced on pp. 442–443. We borrow $P = 300,000$ at $i = \frac{7.8}{12} = 0.65\%$ per month, then after 10 years (240 payments left) refinance it at $i_{new} = \frac{5.8}{12} = 0.4833\%$ per month. We also have 3 discount points in this example as well as a refinancing charge of $F = 500$. Then by the formulas from **Starting a New Loan** on p. 449, we have the following:
• Payment: \( A = \frac{P}{a_{N|i}} = \$300,000 \times \frac{0.0065}{1 - (1 + 0.0065)^{-360}} = \$2,159.61 \)

• Remaining debt:

\[ D_{240} = A \frac{1 - (1 + i)^{-240}}{i} = \$2,159.61 \times \frac{1 - (1 + 0.0065)^{-240}}{0.0065} \]

\[ = \$262,077 \]

• New loan amount:

\[ Q_{240} = \frac{D_{240} + F}{1 - x} = \frac{\$262,077 + 500}{1 - 0.03} = \$270,698 \]

• New loan period: 360 months

• New monthly payment:

\[ C_{240} = \$270,698 \times \frac{0.058 ÷ 12}{1 - (1 + 0.058 ÷ 12)^{-360}} = \$1,588.33 \]

The new monthly payment, $1,588.33, is $571.28 less than the original one. However, by the end of the refinanced loan, the borrower will have paid cumulated interest of

\[ 360 \times \$1,588.33 - \$262,077 = \$309,722, \]

in contrast to the $256,229 (= 240 \times \$2,159.61 - \$262,077) that would have been paid under the original loan. This difference is due to the addition of the 10 years on the refinanced loan. The advantage is that the borrower has extra monthly savings from the reduced payment, for example, to invest (perhaps to earn more than enough to compensate for the increased interest payment). Tables 4 and 5 show other refinancing conditions and their consequences. In all cases, the upper bound for the fee, calculated by \( A\{1 - x\}a_{N|new} - a_{N|old}\) as seen in (9), is well above the actual fee \( F\), so we can conclude that it is worth refinancing as long as we do not mind higher cumulated interest.

**Involuntary Refinancing: Interest-Only Loan**

Although most real-estate mortgage refinancings are long-term refinancings, a special case of an ARM, called an *interest-only loan*, has the structure of short-term refinancing. Typically, the borrower pays only the interest on the principal for a fixed amount of time (usually three to seven years) and then during the remaining loan period pays back the principal following the traditional full amortization method at the prevailing rate at the time of interest reset. As a result, the borrower experiences two interest rates throughout one mortgage, as is the case of the example in Short-Term Refinancing on pp. 450–452. When the second interest rate is higher than the
first one, the monthly payment will jump overnight, which is one reason for the avalanche of house foreclosures these days.

Consider a 30-year interest-only loan of $200,000 such that we pay only the interest during the first five years at 6% per annum (0.5% per month), then switch to an fixed-rate mortgage for 25 years at 9% (0.75% per month). What are the monthly payments and how big is the difference between them?

During the first 5 years, we pay only the interest on the principal, so the monthly payment is $200,000 \times 0.005 = $1,000 for 60 months; then, from month 61, we start paying the principal at 9% throughout 300 payments,
so the payment follows the formula

\[ A = \frac{P}{\alpha_{Ni}} = \frac{Pi}{1 - (1 + i)^{-N}} \]

for \( N = 300 \) and the new payment is \( \frac{\$200,000 \times 0.0075}{1 - (1 + 0.0075)^{-300}} = $1,678.39 \).

So our monthly payment increases by \( $(1,678.39 - 1,000) = $678.39 \).

**Conclusion**

As shown, a lower interest rate alone does not guarantee a profitable refinancing. This is because the NPV of the borrower is not only a function of the interest rate but also depends on other factors, such as the remaining number of payments, the equity built to date, and refinancing fees. As a result, refinancing at a lower interest rate is not always a good idea. Even if the effective rate of interest and the monthly payment decrease after refinancing, the borrower may end up paying more interest throughout the refinanced loan. This is not necessarily bad if there is a tax exemption on the interest. Moreover, the borrower may take advantage of the lower monthly payment and invest the savings for higher return. On the contrary, if there is no tax benefit and/or other investment options, the borrower should keep the old loan. Conclusively, when refinancing, all the variables of the NPV function as well as other investment situations should be considered.

**References**


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