The Mathematics of Amortization Schedules on the TI83

Floyd Vest, Nov. 2011 (Preliminary Version)

Note: TI 83/84 letters are not in italics.

Money wisely invested in a home can provide a secure form of investment. The price of housing has usually risen with general price levels. Homeownership offers tax advantages and provides a form of committed savings and investments. A homeowner may have an added sense of security and achievement.

Once you have decided on an affordable, quality home, the purchasing process begins. We will discuss one of the many elements of this process – the home mortgage, the terms, down payment, monthly payment for principal and interest, and the amortization schedule.

A common kind of mortgage involves a 20% down payment and a 30 year fixed rate, with monthly payments. Shorter term mortgages such as a Fifteen year mortgage are also popular because they save a lot of interest.

In dealing with mortgages, the buyer can use their knowledge of financial mathematics and a calculator such as the TI83. We will do both the mathematics and the TI83 functions and code, and follow the examples in the TI83 manual.

Rounding errors can accumulate, it is best to let you calculator float. We assume that the reader already knows the basics of the present value of an ordinary annuity. Our interests include the capabilities of the TI83 and some common and special mortgage terms.

Example 1. A review of an amortization schedule. On page 14-9 of the TI83 manual we are introduced to a 30 year $100,000 fixed rate mortgage with a RATE of 8.5% with monthly payments for principal and interest of $768.91. (See the Side Bar Notes for a review of the mathematics and code for the TVM Solver for calculating the payment.) We know that part of the payment goes to interest and part is principal repaid. Consider the following amortization schedule.

<table>
<thead>
<tr>
<th>Payment Number</th>
<th>Payment</th>
<th>Interest</th>
<th>Principal Paid</th>
<th>Outstanding Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td>$100,000</td>
</tr>
<tr>
<td>1</td>
<td>$768.91</td>
<td>$708.33</td>
<td>$60.58</td>
<td>99,939.42</td>
</tr>
<tr>
<td>2</td>
<td>768.91</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table1. First row of an amortization schedule

The first Interest payment is \( \frac{0.085}{12} (100,000) = 708.33 \). The Principal paid is \( 768.91 - 708.33 = 60.58 \). The Outstanding balance = 100,000 – 60.58 = 99,939.42.
Example 2. Calculate the balance of the above mortgage immediately after the twelfth payment. To get the numbers in the TI manual, put the calculator in Float/Fix 2. First go to the TVM Solver and put in N = 360, PV = 100000, FV = 0, P/Y = 12, and PMT:END to PMT Alpha Solve to get PMT= -768.91. The bal( function will calculate the loan balance after the twelfth payment using the entries in the TVM Solver.

Code and commentary: On the home screen, 2nd Finance 9 You see bal( Write 12 ) Enter and you see the balance after the twelfth payment of 99244.07. To check this, we calculate on an ordinary scientific calculator the Present value of the remaining 348 payments with interest at \( \frac{.085}{12} \) per period with payments of $768.91 to be

\[
PV = 768.91 \left[ 1 - \left(1 + \frac{.085}{12}\right)^{-348} \right] = $99243.59 .
\]

Notice that this figure for balance is different from that of the TI83. At this stage we will assume that the TI83 has handled the rounding correctly. The PV of the calculator $.48 off.

An accurate amortization program should complete all calculations with no more than one cent error. This requires a program in double precision. (See the References and Exercises.) It is not uncommon for amortization to have sizable error due to rounding.

On the TI83 \( \sum Prn(pmt1, pmt2) \) computes the principal paid during a specified period on an amortization schedule with pmt1 the starting payment, and pmt2 is the ending payment in the range. The pmt1 and pmt2 must be positive integers. To calculate the principal paid through the twelfth payment: Code and commentary: 2nd Finance 0 Enter and you see \( \sum Prn( \text{Write 1 , 12} ) \) Enter you see -755.93 as the accumulated principal paid through the twelfth payment. To check this we calculate $100,000 − 99244.07 = $755.93.

For an amortization schedule, the function \( \sum Int(pmt1, pmt2) \) computes the total interest paid, pmt1 is the starting payment, and pmt2 is the ending payment in the period. Code and commentary: 2nd Finance Alpha A You see \( \sum Int( \text{Write 1 , 12} ) \) Enter You see -8470.99. To check this we calculate 12x768.91 − 755.93 = $8470.99 and the total interest paid through the twelfth payment. (For derivations of the Master TVM formulas used by the TI83 calculations and displayed in the Appendix, see the article in this course entitled, “A Master Time Value of Money Formula.”) See the
Side Bar Notes for a table of different values for the above calculations depending on four different treatments of Float and PMT.)

Example 3. Continuing the TI discussion, we will do a graph and table for the outstanding balance on a 360 month, 8% mortgage, with payment of $800 per month for a loan of $109,026.80. They first put the calculator into fixed decimal mode setting 2, in dollars and cents and into Par for parametric mode. To do this, Code: Mode \( \triangledown \triangleright\triangleright\triangleright \) Enter to set the fixed decimal mode setting 2, then \( \triangledown\triangledown > \) Enter for Par graphing mode.

We will first use the TVM Solver to calculate the amount of the loan. (Make sure you use the Sign Conventions of the TVM Solver.) Code and commentary: \( 2^{\text{nd}} \) Finance Enter 360 Enter 8 Enter 0 Enter \((-\)800 Enter Select P/Y 12 and PMT:END \( \wedge \wedge \) to PV Alpha Solve for a mortgage of $109,026.80. (See the Exercises and Side Bar Notes for the mathematics.) To build the graph of the declining outstanding balance, we will set \( X_{1T} \) and \( T \) from 0 months to 360 months in steps of 12 months and \( T_{1T} \) as bal(T). Code and commentary: Press \( Y= \) to display the parametric \( Y= \) editor. Press \( X… \) to define \( X_{1T} \) as \( T \). Press \( \triangledown \) \( 2^{\text{nd}} \) Finance 9 \( X… \) ) to define \( Y_{1T} \) as bal(T). Set the Window for the graph. Press Window and enter

\[
\begin{align*}
\text{Tmin} &= 0 \\
\text{Tmax} &= 360 \\
\text{Xmin} &= 0 \\
\text{Ymin} &= 0 \\
\text{Tstep} &= 12 \\
\text{Xmax} &= 360 \\
\text{Ymax} &= 125000 \\
\text{xscI} &= 50 \\
\text{Yscl} &= 10000
\end{align*}
\]

On the home screen, turn off all stat plots: Press \( 2^{\text{nd}} \) StatPlot, and turn off all Stat Plots.

On the home screen for the graph, press Trace to draw the graph and get the cursor. You see the graph of the declining balance. To find the balance after the twelfth payment: Press \( 12 \) Enter and you read \( T=12, Y = 108116.04 \). (See the Exercises to check the mathematics.) You are graphing the equation

\[
(2) \quad P = 800 \left[ 1 - \left( 1 + \frac{.08}{12} \right)^{-360-T} \right] 
\]

for \( T = 0 \) to \( T = 360 \), where \( P = \text{bal}(T) \). (See the Exercises.)

To see the amortization table reflecting the graph, press \( 2^{\text{nd}} \) Tblset and Tblstart =0, and \( \Delta \text{Tbl} = 12 \). Then press \( 2^{\text{nd}} \) Table and you will see a table for \( T = X_{1T} \), and \( Y_{1T} \) which is the balanced owed at \( T \). For \( T = X_{1T} = 12 \), you see balance = 108116. For \( T = X_{1T} = 72 \), you see balance = \( Y_{1T} = 102295 \).
To see the graph and table simultaneously, follow the instructions in the TI manual. Notice that the table does not meet the standards of an amortization schedule since it does not display “cents.” (See the Exercises for exploration of graphs and tables for Example 3.)

An amortization schedule with a last odd payment. (We are no longer following the narrative of the TI manual.) It is common to begin the calculation of an amortization schedule with the periodic payment rounded to the nearest cent. As a result, there is likely to be an odd last payment (not the same as the regular periodic payments).

Example 4. Suppose there is a loan of $400 at 6% rate. The borrower agrees to $100 payments each month until the loan is paid off. Find the number of full payments and the final odd payment. First do some calculations

\[
400 = 100 \left[ 1 - \left( 1 + \frac{.06}{12} \right)^{-N} \right] \quad \text{and solve for } N.
\]

With the TVM Solver, I% = 6,

\[
PV = 400, \ PMT = -100, \ P/Y = 12, \ PMT:\text{End, we get } N = 4.05 \text{ payments. This tells us that the loan requires four payments of } \$100 \text{ and a last odd payment. To calculate the last payment, we do}
\]

\[
400 = 100 \left[ 1 - \left( 1 + \frac{.06}{12} \right)^{-4} \right] + L \left( 1 + \frac{.06}{12} \right)^{-5}
\]

so that all payments are discounted to the initial loan value. Calculating, we get \( L = \$5.06 \) as the last payment to be made at the end of the fifth month.

A payment schedule of a savings program with a final odd payment. An investor has a desire to accumulate $10,000 by making $500 quarterly deposits in a bank paying 5% compounded quarterly. Find the number of the payment on which the accumulation first exceeds $10,000. Then calculate the final payment.

You can solve

\[
10000 = 500 \left[ \left( 1 + \frac{.05}{4} \right)^n - 1 \right]
\]

for \( n \), or use the TVM Solver with I% = 5,

\[
PV = 0, \ PMT = (-)500, \ FV = 10000, \ P/Y=4, \ PMT:\text{END} \quad \text{\AAA to } N \text{ and Alpha Solve to read } N = 17.96. \ We \ see \ that \ the \ eighteenth \ payment \ will \ be \ the \ odd \ payment. \ On \ the \ TVM \ Solver, \ we \ calculate \ the \ FV \ of \ 17 \ payments: \ N = 17, \ move \ to \ FV \ and \ Alpha \ Solve \ to \ read \ 9405.53. \ Then \ calculate \ 9405.53 \left( 1 + \frac{.05}{4} \right)^4 + F = 10000 \text{ where } F = \$476.90 \text{ as the eighteenth payment.}
Points on a mortgage. Closing fees and points are charges that are collected in advance. One point is equivalent to one percent of the mortgage amount. For example, one point on the $100,000 loan in Example 1 is $1000, two points is $2000. One point can add a fraction of a percentage point to the effective interest rate you pay on the mortgage. In Example 1, for one point, you can think of borrowing $99,000 for 30 years with monthly payments of $768.91. Using the TVM Solver gives an effective interest rate of 8.61%, an increase from 8.5% of .11 percentage point. Two points give an effective rate of 8.72%, and increase of .22 of a percentage point. (See the Exercises.)

Example 5. Consider a recent (Nov. 2011) advertisement of interest rates for a 30 year fixed rate mortgage on a well known website: RATE 3.250%, APR 3.581%. We know by the spread that points must be involved. Checking the APR:

\[
\left(1 + \frac{.03250}{12}\right)^{12} - 1 = .0329885 = 3.230\%.
\]

Points much be involved. Calculating

\[
100 \left(1 + \frac{.03250}{12}\right)^{12} = P \left(1 + \frac{.03581}{12}\right)^{12}
\]
gives \(P = \$99.67\). \(100 - P = \$.33\).

One point on $100 is $1. There must be 1/3 of a point.

Accuracy of amortization schedules. In a correctly programmed table, except for the periodic payment, the dollar and cents values are for display and calculations are often done to twelve places or more. To illustrate levels of accuracy, we will compare entries in the balance owed column from four sources: a table that is supposed to be accurate to within less than one cent, a nice looking table from a well known website, the bal( function of the TI83, and a scientific calculator. The loan is for $1000 with 12 monthly payments of $88.62 at a rate of 11.50%.

<table>
<thead>
<tr>
<th>TI83</th>
<th>Accurate Table</th>
<th>Website Table</th>
<th>Scientific Calc</th>
</tr>
</thead>
<tbody>
<tr>
<td>bal(1)=920.96</td>
<td>920.97</td>
<td>920.97</td>
<td>921.02</td>
</tr>
<tr>
<td>bal(5)=597.17</td>
<td>597.17</td>
<td>597.19</td>
<td>597.23</td>
</tr>
<tr>
<td>bal(11)=87.72</td>
<td>87.72</td>
<td>87.77</td>
<td>87.78</td>
</tr>
<tr>
<td>bal(12)=.0625</td>
<td>0 (after odd payment of $88.56)</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 2. Accuracy comparison. (The TI83 was in Float/Fix 2. It gets different values for different Float and PMT. It is best if in Float, to put in manually the PMT rounded to cents as in 88.62.)

Notice that the TI83 bal( and the Accurate Table were within one cent. If the amount of the payment is calculated in Float by the TI83, PMT = 88.615053….The $88.62 payment is the result of rounding up. This accounts for the bal(12) = -.06, more accurately -.0624, for the TI83, and the smaller odd payment of $88.56. In these examples, the scientific calculator and website were off by as much as 5 cents. (See the Exercises.) For the website table, it is programmed to always give a final balance of $0 and a final payment equal to the periodic payment.
Two methods for building an amortization schedule. The most intuitive method is that used in building the table in Example 1. (It will be left as an exercise.) This could be called the recursive method because calculations for a row depend on the previous row.

The other method, the closed form equation, calculates the balance owed for Line \( n+1 \) (payment \( n+1 \), immediately after payment \( n \)) with Balance \( = P \left[ \frac{1-(1+i)^{-(t-n)}}{i} \right] \)

where \( P \) is the periodic payment, \( i \) is the interest rate per period, \( t \) is the total number of periods. From this balance, the interest can be calculated and subtracted from \( P \), to give the amount that went to principal. Both methods give accurate tables if programming calculations are done with double precision. (See the Exercises.)

**Side Bar Notes:**

For Example 1: To calculate the monthly payment for Example 1 on the TVM Solver: Code and commentary: 2nd Finance Enter Put in N = 360, I% = 8.5, PV = 100000, FV = 0, P/Y = 12, PMT:END. Go back up to PMT and Alpha Solve. Your read PMT = -768.91348… Round PMT to $768.91 per month. You are solving the formula

\[
100,000 = \text{PMT} \left[ \frac{1-(1+\frac{.085}{12})^{-360}}{\frac{.085}{12}} \right].
\]

In Example 3, you are solving PV = 800 \( \left[ \frac{1-(1+\frac{.08}{12})^{-360}}{\frac{.08}{12}} \right] \) for the mortgage amount of $109,026.80.

Record low mortgage interest rates, Nov. 2011. “Rates for 30-year fixed mortgages are hovering at 4%, and 15-year fixed loans can be had for 3.5% or less, the lowest in more than 50 years” (Forbes, Dec. 2011).

**Accuracy on the TI83 depending on Float and PMT**

<table>
<thead>
<tr>
<th>TI manual p 14-9</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>bal(12) =99244.07</td>
<td>99244.07</td>
<td>99244.07</td>
<td>99244.08097</td>
<td>99244.0375…</td>
</tr>
<tr>
<td>( \sum \text{Pr}(1,12) ) =-755.93</td>
<td>-755.93</td>
<td>-755.93</td>
<td>-755.9190272</td>
<td>-755.969624978</td>
</tr>
<tr>
<td>( \sum \text{Int}(1,12) ) =8470.99</td>
<td>-8470.99</td>
<td>-8470.99</td>
<td>-8471.000973</td>
<td>-8470.999305…</td>
</tr>
</tbody>
</table>

A: In Float/Fix2, Let TVM Solver calculate PMT. B: In Float/Fix2. Put in manually PMT=-768.91 Enter. C: In Float. Put in manually PMT=-768.91 Enter. D: In Float,
Let TVM Solver calculate \( PMT = -768.91348\ldots \) Notice the differences. For amortization PMT should be \(-768.91\) which is what the borrower pays.

Searching for housing? Using Zillow, a free app from App Store, you can search by price or monthly rent for housing. Uses a GPS locator. You can access the value of any home. Search classes include houses for sale, houses recently sold, number of bedrooms, locations, and so on.

Exercises.

1. (a) For the mortgage in Example 1, calculate the APR and the total interest paid in 30 years. Use both mathematical formulas and financial functions such as eff for the APR. (b) For an 8\%, 15 year mortgage, calculate the payment and the total interest paid. How much interest is saved by the 15 year mortgage? (c) For a more current mortgage rate of 4.5\% (Nov. 2011) and 30 years, calculate the payment and total interest paid. (d) For a 15 year mortgage at 4\%, calculate the payment and the total interest paid. (e) Do the above calculations for interest rates which you find currently. For example, check bankrate.com.

2. Use the numbers in Examples 1 and 2 to build the amortization schedule entries for the 13\(^{th}\) payment including the beginning balance, payment, the interest paid, the principal paid, the final balance, the total interest paid through the 13\(^{th}\) payment, the total principal accumulated through the 13\(^{th}\) payment.

3. For Example 1, if two points are charged, what is the effective nominal rate and the effective APR? By how much is the APR increased?

4. Use a convenient program and print a complete amortization schedule for the mortgage in Example 1. One way to find a program is do an internet search for amortization schedule. If you see Amortization Calculator http://amortization.bankrate.com/. You can use it.

5. Experiment with the effect of rounded payments for the mortgage in Example 1. Put you calculator in Float mode. (a) Let the TVM Solver calculate the payment. Report. (b) Round up and calculate PV for 360 payments. (c) Round down and calculate PV. Discuss. You can read about accuracy on the TI83 on p. B-10 of the manual. (d) For each rounded payment, calculate the odd last payment.

6. For the mortgage in Example 1, with RATE = 8.5\%, if the APR is reported as 9.5\%, how many points are charged and what is the dollar amount? Use both mathematical formulas and TVM functions.

7. (a) Using the TI83 scientific calculator pad and mathematical formulas, recalculate the Scientific Calculator column in Table 2. Put your calculator in Float. Discuss. (b) Use both mathematical formulas and TVM functions to calculate the last payment suggested by the bal(12) = -.0625.
8. For the mortgage in Example 1, write out the general steps for using (the closed form method) the balance after the kth payment to construct the entries in the row for the k+1 payment. You should describe four or five steps. Include the TVM code and mathematical formulas. Calculate the first two rows and the last two rows with a scientific calculator and with TVM functions.

9. For the mortgage in Example 1, write out the general steps for calculating the row for the k+1 payment by the recursive method. Calculate the first two rows and the last two rows with a scientific calculator and with TVM functions. Discuss accuracy.

10. A general mathematical structure of an amortization schedule. (From a paper by John A. Winterink, “Amortization of a Consumer Loan,” Fall, 1991, but the author could not find the name of the journal.) Consider the following definitions and theorems. In proving a theorem, use only definitions and previously proven theorems, and basic algebra.

Definitions: $A$ = original balance of loan. $P_k$ indicates payments for n payments. $B_k$ is the balance immediately after the kth payment. Let the principal reduction in $P_k$ be called $R_k$ where $R_k = B_{k-1} - B_k$. The interest in $P_k = P_k - R_k$.

The interest in $P_k = iB_{k-1}$ where $i$ is the periodic interest rate. $A = \sum_{K=1}^{n} R_k$. Total interest paid = $\sum_{K=1}^{n} P_k - \sum_{K=1}^{n} R_k$.

Prove that $A = \sum_{K=1}^{n} (B_{k-1} - B_k)$. Prove that total interest can be expressed in the following ways: Total interest paid = $\sum_{K=1}^{n} P_k - \sum_{K=1}^{n} R_k$.

Total interest paid = $i \left( \sum_{K=0}^{n} B_K \right) = iA + i \sum_{K=1}^{n} B_K = iA + i \sum_{K=1}^{n-1} B_K = i \sum_{K=0}^{n-1} B_K$

Total interest paid = $\sum_{K=1}^{n} P_k - A$.

If $P_K = P$ for K = 2,3,…n-1, Total interest paid = $P_1 + P(n - 2) + P_n - A$.

Total interest paid = $i \sum_{K=1}^{n} B_K + (P_1 - R_1)$. (b) Express PV = A in terms of $P$ (regular payments P), $i$, and n, as a mathematical formula. Solve for $P$. Solve for n. Solve for the odd last payment $P_n$.

11. (a) The pay off Balance immediately after the kth regular payment $P$, where $P_n$ is the odd last payment is

\[ \text{Balance} = P \left[ \frac{1-(1+i)^{-k}}{i} \right] + P_n (1+i)^{-n} \]

Show that if $P_n = P$, this expression is

\[ \text{Balance} = P \left[ \frac{1-(1+i)^{-k}}{i} \right] \]  

(b) For the original amount of the mortgage A,
\[ A = P \left(1 - (1 + i)^{-(n-1)} \right) + P_n (1 + i)^{n}. \]

Solve in general for \( P \), \( P_n \), and \( n \). For methods of solving numerically for \( i \), see Unit 6: Solving Financial Formulas for Interest Rate, in this course.

12. By any method, calculate and print an amortization schedule for the mortgage in Table 2.

13. A home can be a good investment. For existing homes, the median price in 1978 of $48,700 increase to $127,200 by 1997. What was the average annual rate of increase in home values?

14. If \( n = 1, 2, 3, \ldots \) then \( S = R \left[ \frac{(1+i)^n - 1}{i} \right] \). Prove this formula by mathematical induction on \( n \).

15. (a) For a $65,000, 30 year mortgage at 10\%, if a friend, as a favor on the day of closing, makes a payment for the buyer, how much interest is saved? (b) For the above original mortgage, if the buyer makes half a payment every two weeks for 26 payments per year, how much interest is saved?

16. For a mortgage of $70,000, RATE = 7\%, payment = $465.71 per month, for 30 years, write a program to use the TI83 to build an amortization schedule with each row reading Beginning Balance, Payment, Amount to Interest, Amount to Principal, and Ending Balance. Show your program and the first two rows and the last two rows of the table. You can use the TVM functions and the Table function. Put the calculator in Float, 2.

17. Use the TI83 Solver, p. 2-10, to solve for the annual rate \( I \) for a mortgage of $70,000, \( N = 360 \), payments are $465.71 per month. Show your code, formula, and answer.

18. (a) Use Formula 2 in Example 3 to calculate \( \text{bal}(12) = $108,116.04 \). (b) For Example 3, do a graph of the accumulations of principal repaid. (c) Do a graph of the accumulations of interest paid. (c) On the same screen do a graph of the accumulating principal paid and the declining loan balance. (e) Examine the graph to show that the total principal repaid is the initial amount of the mortgage. (f) Do the tables.

19. From the Denton Record Chronicle, Dec. 6, 1987. “Many lenders use points as a marketing device. They may quote an 11.25\% rate on a fixed-rate mortgage when the going rate is 12\%. The 12\% mortgage comes with just two points while the 11.25\% requires six points up front.” Consider a $60,000 new loan, for 30 years. Which is the best deal? What is the effective interest rate for each?
Answers to Exercises:

1. (a) Payments = $768.91. APR = \(1 + \left(1 + \frac{.085}{12}\right)^{12} - 1 = 8.83909\ldots = 8.84\%\) or with the
TVM function \(\text{Eff}(8.5, 12) = 8.83090\ldots = 8.84\%\). Total interest paid at the end =
\(360(768.91) - 100,000 = 176,807.60\). (b) On the 15 year mortgage, payment =
$955.65. Total interest = 180(955.65) – 100,000 = $72,217. Less interest by $97,590.60.

2. For the 13\textsuperscript{th} payment, the beginning balance = $99,244.07. Payment = $768.91.
Interest = $702.98. Principal = $65.93. Total principal accumulated = $821.85. Total
interest paid = $9173.98.

3. With two points, the effective nominal rate = 8.72\%. The APR = 9.08\%. For 8.5\%
rate, the APR = 8.84\%. The increase in APR = .24 percentage point.

5. (a) In Float, the TVM solver gets \(\text{PMT} = -768.91348\ldots\). (b) Rounding up with
payment = $768.92, \(PV = $100,000.847\). (c) Rounding down, payment = $768.91,
\(PV = $99999.54695\). (d) With payment rounded down to $768.91, the final payment =
$774.67.

6. For the APR = 9.5\%, the nominal rate = 9.1\%. Calculating with \(PV =
\[
768.91 \left[ 1 - \left(1 + \frac{.091}{12}\right)^{-360} \right] \] 
gives \(PV = 94,713.37\). \(100,000 - 94,713.37 = 5,286.63\) in
points, or 5.29 points. Using \(\text{Nom}(9.5, 12)\) gives 9.1.

7. (b) Using \(1000 = 88.64 \left[ 1 - \left(1 + \frac{.115}{12}\right)^{-11} \right] + L \left(1 + \frac{.115}{12}\right)^{-12}\) gives the last payment
\(L = 88.32\)

8.

<table>
<thead>
<tr>
<th>Payment Number</th>
<th>Beginning Balance</th>
<th>Payment</th>
<th>Interest</th>
<th>Principal</th>
<th>Ending Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$100,000</td>
<td>$768.91</td>
<td>$708.33</td>
<td>$60.58</td>
<td>$99,939.42</td>
</tr>
<tr>
<td>2</td>
<td>99,939.42</td>
<td>768.91</td>
<td>707.90</td>
<td>61.06</td>
<td>99,878.41</td>
</tr>
<tr>
<td>359</td>
<td>1527.30</td>
<td>768.91</td>
<td>10.82</td>
<td>758.09</td>
<td>769.20</td>
</tr>
<tr>
<td>360</td>
<td>769.21</td>
<td>768.91</td>
<td>5.45</td>
<td>763.46</td>
<td>5.75</td>
</tr>
</tbody>
</table>

(The $5.75 is due to rounding down the payment.)
10. (b) \[ n = \frac{\log \left( \frac{1 - Ai}{P} \right)}{1 + i}, \quad P_n = A - P \frac{\left[ 1 - (1 + i)^{-(n-1)} \right]}{i} \]

11. (b) \[ n = \frac{\log(1 + i)}{\log \left( \frac{Ai - P}{P(1 + i) + iP_n} \right)} \]

13. \(48,700(1 + i)^{19} = 127,200, \ i = 5.18\% \) average per year appreciation.

15. (a) \(PV = $65,000, \ n = 360, \ i = \frac{.10}{12} \), \(PMT = $570.42\), Total interest = $140,351.20
\[ PV = 65,000 - 570.42 = 64,429.58, \ N = 341.57 \text{ payments to pay off.} \]
Interest = 341.57(570.42) - 64,429.58 = 130,408.48.
Savings in interest = 140,351.20 - 130,408.48 = $9,943.42.  (b) Originally, \( \text{Total interest} = 360(570.42) - 65,000 = $140,351.20 \). For the biweekly,
\[ \text{PMT} = \frac{570.42}{2} = $285.21. \text{ } 65000 = 285.21 \frac{1 - \left( \frac{10}{26} \right)^{-N}}{\frac{10}{26}}, \ N = 544.94 \text{ payments.} \]
Total interest = 544.94(285.21) - 65,000 = $90,422.34.
Interest saved = 140,351.20 - 90,422.34 = $49,928.86.

16. \(PV = $70,000, \ I\% = 7, \ N = 360, \ PMT = $465.71\)

<table>
<thead>
<tr>
<th>Month</th>
<th>Beginning Balance</th>
<th>Payment</th>
<th>Interest</th>
<th>Principal Reduction</th>
<th>Ending Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$70,000</td>
<td>465.71</td>
<td>408.33</td>
<td>57.38</td>
<td>69,942.62</td>
</tr>
<tr>
<td>1</td>
<td>69,942.62</td>
<td>465.71</td>
<td>408.00</td>
<td>57.71</td>
<td>69,884.91</td>
</tr>
<tr>
<td>2</td>
<td>69,884.91</td>
<td>465.71</td>
<td>408.00</td>
<td>57.71</td>
<td>69,827.20</td>
</tr>
<tr>
<td>359</td>
<td>925.44</td>
<td>465.71</td>
<td>5.40</td>
<td>460.31</td>
<td>465.13</td>
</tr>
<tr>
<td>360</td>
<td>465.13</td>
<td>465.71</td>
<td>2.71</td>
<td>463</td>
<td>.13</td>
</tr>
</tbody>
</table>

19. For $60,000 at 12%, the payment = $617.17. with two points, the effective loan is $60,000 - 1200 = $58800. This makes I\% = 12.27\%. For $60,000 at 11.25%, payment = $582.76. with six points the effective loan is $60,000 - 3600 = $56400. This make I\% = 12.06\%. So the 11.25\% with six points is the better deal, if you can afford six points, $3600. \text{Eff(12.27,12) = 12.984\%.  Eff(12.06,12) = 12.749\%.}
References

Federal Reserve Board, Official Staff Commentary on Regulation Z, Truth In Lending.
For the Truth in Lending Act on the internet, do a search for Truth in Lending.

In this course, see the following related articles:

“A Master Time Value of Money Formula” for derivations of the TI83 basic TVM
formulas found in the Appendix to the TI manual. See other articles, including some in
Unit 5: Investing in Real Estate.

TI83 Graphing Calculator Guidebook, Texas Instrument, 1996. (For a copy of the TI83
manual, see www.ti.com/calc.)

Teachers’ Notes:

For a free course in financial mathematics with emphasis on personal finance, see
COMAP.com. Click on the new course and register. COMAP will send you a password.
Simply click on an article in the annotated bibliography, download it, and teach it.
Unit 1: The Basics of Mathematics of Finance Unit 2: Managing Your Money
Unit 3: Long-Term Financial Planning Unit 4: Investing in Bonds and Stocks
Unit 5: Investing in Real Estate Unit 6: Solving Financial Formulas for Interest Rate

For instructions in BASIC and a flow chart for building an amortization schedule,
see in this course, Kasting, Martha, Module 366, Concepts of Math for Business: An
Introduction to Basic Language for Computers, 1982. Chapter 6 deals with Compound
Interest, Savings Plans, Sinking Funds, and A Mortgage Schedule.