Example 1: We will build a portfolio of bonds so that whether interest rates rise or fall, the portfolio can pay off an obligation of $10,000 in two years. The duration of the $10,000 obligation on two years is two, since it due in two years. (See the Side Bar Notes and References.)

The portfolio manager is considering two bonds:

A one year $1000 bond paying 1040.06 at maturity. If interest rates are 9.5%, the one year bond is purchased for $949.83. It has a duration of one year.

A three year $1000 annual pay bond with $75 in coupons and paying $1000 + $75 = $1075 at maturity. At 9.5%, the three year bond can be purchased for $949.83. Its duration is 2.8 years. (See the Side Bar Notes and the References).

Since the prevailing interest rate is 9.5%, to meet the $10,000 in two years requires and investment of $10,000/(1+.095^2) = $8340.11. One proposed solution is to invest part in the one year bond and part in the three year bond. If immunization is used, weighting the parts that are invested in the two different bonds by $W_1$ for the one year bonds and $W_3$ for the three year bonds would give

\[(1) \quad W_1 + W_3 = 1 \quad \text{and} \quad (W_1 \times 1) + (W_3 \times 2.8) = 2 \quad \text{where} \ 1, \ 2.8, \ \text{and} \ 2 \ \text{are durations.}\]

Solving gives $W_1 = .4444$ and $W_3 = .5556$. Applying these weights

$W_1 \times 8340.11 = .4444 \times 8340.11 = $3706.35$ is used to purchase one year bonds.

$W_3 \times 8340.11 = .5556 \times 8340.11 = $4622.77 is used to purchase three year bonds.

At current 9.5% market rates, the number of one year bonds purchased is 3706.35/949.83 = 3.9021 bonds. The number of three year bonds purchased is 4633.77/949.82 = 4.8786 bonds.

To meet the two year obligation of $10,000 we will build Table 1 assuming interest rates remain at .095 = 9.5%.

Table 1. Two year value of bonds at 9.5%:

At two years, the value of the one year bonds: $1040.06 \times 3.9021 \times (1+.095) = $4443.97 .

Investment in three year bond: Coupon at end of year one: $75(4.8786)(1.095) = $400.66 .

Coupon at end of year two: $75(4.8786) = $365.90 .

Selling the three year bond at the end of year two, the value at the end of year two is $[1075 \times 4.8786]/1.095 = $4789.49 .

At the end of year two, the three year bonds pay $400.66 + $365.90 + $4789.49 = $5556.04 .
The investment in the two bonds pays $4,443.97 + $5,556.04 = $10,000 which was the obligation to be paid in two years by the bond portfolio.

Let us assume that interest rates rise to 10% and build Table 2.

Table 2. Two year value of bonds at 10%:

At two years, the value of the one year bond is $10,400.06 \times 3.9021 \times (1 + 0.10) = $4,464.26.

Investment in three year bond: Coupon at end of year one: 75(4.8786)(1.10) = $402.48.

Coupon at end of year two: 75(4.8786) = $365.90.

Selling the three year bond at year two, for the value at the end of year two: \[\frac{1075 \times 4.8786}{1.10} = 4767.72\].

At the end of year two, three year bond pays $402.48 + $365.90 + $4767.72 = $5,536.10.

The investment in the two bonds pays $4,464.26 + $5,536.10 = $10,000 which was the obligation to be paid in two years by the bond portfolio. (Excuse the rounding.)

Let us assume that interest rates drop to 9% and build Table 3.

Table 3. Two year value of bonds at 9%:

Value of one year bond at year two: $10,400.06 \times 3.9021 \times (1 + 0.09) = $4,423.68.

Investment in three year bond: Coupon at end of year one: 75(4.8786)(1.09) = $398.83.

Coupon at end of year two: 75(4.8786) = $365.90.

Selling the three year bond at year two, for the value at the end of year two: \[\frac{1075 \times 4.8786}{1.09} = 4811.46\].

At the end of year two, three year bond pays $398.83 + $365.90 + $4811.46 = 5,576.19.

The investment in the two bonds pays $4,423.68 + $5,576.19 = $10,000 which was the obligation to be paid in two years by the bond portfolio. (Excuse the rounding.)

Overall Percentage Return.

We consider the overall percentage return on the above three year bond at 9.5% and sold at year two at 9% in terms of the Time effect + Yield change effect + Sum of coupons + Interest on coupons. We assume that coupons are reinvested at 4%.

Time effect: We assume that for the first two years the interest rate is 9.5%. At time zero, the bond is worth $949.83. At two years, it is worth $1075/(1.095) = $981.74. The time effect = 981.74 – 949.83 = $31.91.

Yield change effect. We assume the yield drops from 9.5% to 9%. At Year two, at 9% the bond is worth $986.24. At 9.5% it was worth $981.74. When sold for $986.24, the yield change effect = 986.24 – 981.74 = $4.50.

The sum of coupons = 2 x 75 = $150.
The interest on coupons reinvested at 4%. With two $75 coupons invested at 4%. The future value = $153. The interest on coupons = $153 – 150 = $3.

Overall dollar amount = Time effect + Yield change effect + Coupons + Interest on coupons

\[ = \$31.91 + \$4.50 + \$150 + \$3 \]

\[ = \$189.41 \]

The price a time zero = $949.83. The Overall percentage of return = 189.41/949.83 = 19.9%. (Sharpe, page 394)

Side Bar Notes:

Duration of the above three year bond. Interest rate 9.5%.

<table>
<thead>
<tr>
<th>Time</th>
<th>Cash Flow</th>
<th>PV Factor</th>
<th>Present Value</th>
<th>Present Value × Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>75</td>
<td>.9132</td>
<td>68.49</td>
<td>68.49</td>
</tr>
<tr>
<td>2</td>
<td>75</td>
<td>.8340</td>
<td>62.55</td>
<td>125.10</td>
</tr>
<tr>
<td>3</td>
<td>1075</td>
<td>.7617</td>
<td>818.83</td>
<td>2456.49</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>949.87</td>
<td>2650.08</td>
</tr>
</tbody>
</table>

Duration = 2650.08/949.87 = 2.8 years. What variables determine duration? Would the above $1000 bond paying 15% and priced at 9.5% have a greater or smaller duration? What would happen to the price of this bond if interest rates rose by one percentage point? See the article in this course on duration.

Price of the three year bond at 9.5%.

Price = 75/1.09 + 75/1.0952 + 1075/1.0953 = $949.83 What variables determine the price of a bond? Do they determine duration?

(See the articles in this course on bond duration and bond pricing.)


Daniel, James W. and Leslie Vaaler, Mathematical Interest Theory, Prentice Hall, 2007, p. 437, 438 uses calculus to derive immunization and gives the assumptions about the yield curve. On page 438, they conclude that the derivative \( S'(i) \) must be zero is equivalent to the duration of the assets equaling the duration of the liabilities. This book was written by faculty members in the Mathematics Department at the University of Texas and is taught in that department.

for which the duration is $D_M = 0$, and $y$ dollars invested in a two year zero coupon bond with a duration $D_Z = 2$. There is a $1100$ obligation due in one year which has a duration of $D_O = 1$. Transactions are at 10%. We let $P(i) = \text{Present value of (Assets – Obligation)}$. Solving gives $x = 500$ and $y = 500$. We calculate the first derivative $P'(i) = 0$ and get $0 = \frac{-2y}{1.1} + \frac{1000}{1.1}$.

Manipulating gives

$$0 + 2 \left( \frac{y}{x+y} \right) = 1.$$ For the weighting, we have $W_M = \frac{x}{x+y}$ and $W_Z = \frac{y}{x+y}$. Thus we have

$$0 \left( \frac{x}{x+y} \right) + 2 \left( \frac{y}{x+y} \right) = 1 \text{ or } D_M W_M + D_Z W_Z = D_O \text{ and } W_M + W_Z = 1 \text{ (as above in the first example in this article). This answer has been verified to immunizes the portfolio to meet the Obligation. The derivation found a local minimum point for $P(i)$ by setting $P'(i) = 0$, and checking $P''(i)$ for positive. $P'(i)$ give the slope of the curve at the local minimum and $P''(i)$ gives the rate of change of slope which is positive at a local minimum. You can draw pictures to verify this. For $P'(i)$ and $P''(i)$ you could use the $dy/dx$ function on the TI84. If you know calculus, you could do this derivation. Although this book has been around a long time, it is still in print.}

A ladder of bonds. Some investors prefer a ladder of bonds to a bond mutual fund because you know exactly what money you will receive and when. With a bond fund, you have the uncertainty of NAVs going up and down with the uncertain interest rates of the market. You can build a ladder of bonds with “deferred maturity” bond funds which hold bonds that mature at the same time. At maturity, these ETFs distribute assets (value at maturity of bonds and reinvested dividends minus fees) back to shareholders. (Money, Dec. 2014). This provides safety, diversification, and convenience for a fee. For example, Guggenheim Bullet Shares 2016 High Yield Corporate Bond ETF (BSJC) yields 4.1% and terminates in two years in 2016. See the internet for “deferred maturity” bond funds.

 Aren’t bond prices volatile? Compared to stocks, when bonds prices drop in a year, they drop a lot less. The worst year stocks was -43%. The worst year for bonds was -8% (Money, Dec. 2014).

The yields of different types of stocks at different times. Vanguard Value Index (VIVAX) has beaten the S&P 500 over the past 15 years while falling less in down months. The American Century Value (TWVLX) has outpaced the S&P 500 by more than three percentage points annually.

Arbitrage is the ability to make a riskless profit. For example, say you see the following exchange rates: One GPB (British Pound Sterling) buys 130.47 JPY (Japanese Yen). One GBP buys 1.466 USD (US Dollar). One USD buys 89.1 JPY. You can use one GBP to buy 1.466 USD. Use the 1.466 USD to buy 1.466 × 89.1 JPY. Use the JPY to buy GBP. Show that you end up with 1.0012 GPB, and a profit of .0012 GPB = .0012(1.466) = $0.0017592. Do this with one million GPB ($1,466,000) and you make $1759.20 . Do this 100 times a day and you
profit $175,920 per day. (What is Financial Mathematics? By Tim Johnson at plusmaths.org)

See the internet for the current GBP to USD.

In derivative construction, the basic principle is that no arbitrage profits can be made. There are mathematical theorems which tell how to do this, but they have been misapplied to the extent of causing the international mortgage crisis. (See Wikipedia.org.)

Yield curve. Much of the discussion of derivations and problems of immunization is associated with assumptions about the yield curve. Historically, the yield curve has changed from time to time and at times it is referred to as flat, other times inverted, and at time normal. A widely used yield curve in based on Treasury zero coupon rates out to 30 years, often called spot rates or zero coupon rates. Mathematically, the spot rate $r_t$ is the annual effective rate earned by money invested from $t = 0$ for a period of $t$ years. For current Treasury yields, search Daily Treasury Yield Curve Rates. To see graphs, search Images for yield curve graph today. To read the graphs, click on the image. Read numbers on the vertical axis. Sketch a graph of a normal curve, a flat curve, and an inverted curve. How would you describe the yield curve for your date? (See the references.)

Search for Wolfram mathworld calculator. Click on Examples by topic. Click Money and finance. You will see: bond pricing formula, United States 30 year Treasury bond, treasury yield curve, mortgage, Present value, currency, Historical use of money. Click on What you want to know or calculate: Type in duration of a bond, try immunization. Try topics of interest in other parts of Wolfram.

Fed Funds rate was 0.25% at the time of this writing and it is expected to rise from this historically very low rate. The Fed Funds rate is the rate that banks charge each other for overnight loans. The Federal Open Markets Committee (FOMC) sets the rate each quarter. Most members of the FOMC expect the rate to be up to 1.25% by the end of 2015, 3% by the end of 2016, and 3.75% by the end of 2017. (Kiplinger’s Personal Finance, 12/2014) See “How the federal funds rate affects finance” at bankrate.com for the effect of rising Fed funds rates on CD, and mortgage rates.

Science and engineering jobs are expected to grow at more than double the rate of the overall U.S. labor force through 2018 (Fortune.com).

To close to retirement for 70% in stocks. In 2007, on the eve of the 38.4% meltdown in stocks, almost half of workers age 56 to 65 had 70 percent or more of their 401k in stocks. With $1M, invested how much was lost? In the past decade, the US has been hit with two of five of the worst bear markets in the past century. What percentage gain is required to reach the former $1M? Of workers surveyed, 25% said the crisis hurt but I’ll recover, 34% said I may never retire. From 2000 to 2010, the percentage of workers who postponed retirement to age 65 grew from 19% to 30%. It is recommended that for those in their sixties, cap stocks to 50% of portfolio. In 2007, of 60-year-olds 30% had 80% or more in stocks. By postponing Social Security from age 62 to age 70, there is $78,500 in added lifetime benefits. Your checks will be 76% bigger. (Money, Oct. 2010, Oct. 2013) To verify this would be an interesting problem in financial mathematics.
Free course in computational investing is offered as a Coursera Class. Of 53,205 students who enrolled, 2532 passed the course. Carnegie Mellon University says they flag the top students and pass this information on to employers. (Monry, May 2013) On the internet, see Coursera and edX, or MOOC courses.

Student loans. Of the $337 Billion in student loans, $89 billion are overdue, $39 billion are in default. The government has new options. To qualify your payment must be 10% to 20% of your discretionary income. Consider the advantages of “deferral.” Making 120 on-time monthly payments while doing qualifying volunteer work may earn “forgiveness.” (Money, 2014) See the article in this course on working your way through college. Twelve percent do.

A cash-in refinance. For a $200,000 loan balance, refinance at 4.4%, gives monthly payments of $1000 and total interest payments $160,500. Do a cash-in refinance. Pay $25,000 and refinance $175,000 for 15 years, gives monthly payments of $1285 and total interest payments $56,400. Do the calculations to verify this. Discuss the alternatives.

Exercises: Show your work. Label answers, numbers, and variables. Conduct discussions in complete sentences.

#1. An investor has an obligation of $100,000 in two years. He is considering bonds to pay the obligation in two years. One is a one year $1000 bond maturing at $1080. The one year bond is to be renewed to contribute to the obligation. The other bond is a three year annual pay bond paying coupons of $80 maturing at $1080. It is to be sold in two years at the then prevailing rate to contribute to the obligation. (a) At 10%, what is the price of the one year bond? At 10%, what is the price of the three year bond? (b) Calculate the duration of each of the obligation and the bonds. (c) Apply the durations for weights (or proportions) $W_1$ and $W_3$ on the one year and three year bonds for an immunized portfolio to meet the obligation. (d) At 10%, what is the present value of the $100,000 due in two years? How much is spent on each type of bond? How many of each are acquired? (e) Show that the portfolio is immunized for remaining interest rate of 10%, for 9.5% interest rate, and 10.5% interest rate.

#2. Explain how duration of 2.8 years for the three year bond is used to measure the bond price change with an interest rate change. Demonstrate this effect by calculating the bond prices. Calculate the duration for the two year debt of $10,000 at 10%.

#3. The duration for a bond fund is the weighted durations for the bonds it holds. Assuming a seven year average maturity and a one time change in interest rates across maturities. For an index bond fund yielding 2.1% if rates rise 1 percentage point, NAV drop 3%; for a 2 points rise, NAV drops 8%; for a 5 points rise, NAV drops 23%. In any case, the annualized return after seven years is about 2%. (Money, Jan./Feb. 2015) Discuss.

#4. Make an example of a three year bond and calculate and calculate the Overall percentage return at year 2, based on your example of a drop in interest rate.

#5. Derive that the duration of a zero coupon bond is the length of life of the bond.
#6. When Bernanke toughened his talk in mid-June 2013, the yield on 10-year Treasury notes has vaulted from 1.63% in early May to over 3.5% by July dropping prices by almost 7% (Fortune, July 22, 2013). Verify this.

#7. One source says that saving $200 per month from age 22 to age 67 at 8% generates $1M. They say saving $263 a month from age 27 to age 65 at 8% generates $1M. How did they get these results? What did they use for 8%? How many years or months did they use. Did they use payment at the beginning of each month or at the end of each month? You can probably make this work on the TI 84 TVM Solver but it takes some intermediate calculations. If they wait until age 47, do they have to save $1637 per month and for age 57, save $5376 a month. Buy the 4% rule, what retirement savings do you need? Give you assumptions.

#8. Does the immunization in Example 1 work with large changes in interest rate such as down to 1% and up to 20%?

**Answers:**

#3. With seven years average maturity, about half the bonds would mature and be reinvested at 2.15% + 1%, 2.15% + 2%, 2.15% + 5%. In any case, after seven years, the average yield would be about 2%, even with a 5 points rise and a 23% drop in the value of your investment. The larger the decline, the higher the new investment rate. What would be the NAV?

**References** in this course: See these references for derivations, proofs, and calculations.

The Mathematics of Bond Pricing and Interest Rate Risk.

Bond Pricing Theorems.

Bond Duration.

General Annuities and Bond Prices Between Coupon Dates.

Bond Convexity

The Effect of Interest Rate Increase in a Low Interest Rate Environment.

Income Taxes on Zero Coupon Bonds.

Yield Curves.

Other references:


For a free course in financial mathematics, with emphasis on personal finance, for upper high school and undergraduate college, see COMAP.com. Register and they will e-mail you a password. Simply click on an article in the annotated bibliography, download it, and teach it. Unit 1: The Basics of Mathematics of Finance, Unit 2: Managing Your Money, Unit 3: Long-Term Financial Planning, Unit 4: Investing in Bonds and Stocks, Unit 5: Investing in Real Estate, Unit 6: Solving Financial Formulas for Interest Rate. For about thirteen more advance or technical articles, see the UMAP Journal at COMAP. The last section is Additional Articles on Financial Mathematics or Related to Personal Finance. In all, there are about eighty articles.