

## Solving Annuity Formulas for Interest Rate – Iterative Routines on the TI 83/84

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Basic to mathematics of finance is the formula for the sum of an ordinary annuity. The formula in traditional terminology is

$$(1) \quad S = R \left[ \frac{(1+i)^n - 1}{i} \right], \text{ where } S \text{ is the sum, } i \text{ is the rate of interest per compounding}$$

period,  $R$  is termed rent which is generic for payment or withdrawal per compounding period,  $n$  is the number of compounding periods (Bowen, 1976). (See the manuals for the TI 83/84 for notation and annuity formulas for the TVM Solver. See in this course, Vest, Floyd, "A Master Time Value of Money Formula," for derivations of the annuity formulas as used in the TVM Solver.)

Most students are aware that they can solve explicitly for all variables in annuity formulas except for  $i$ . (See the Exercises.) It is well known that there are iterative programs for solving numerically for  $i$ .

#### **Programming the TI83/84 to compute $i$ in the formula for the sum of an ordinary annuity (Newton's Method).**

(We note that the original Formula 1 is undefined at  $i = 0$ .) Newton's method uses some differential calculus. Many students studying financial mathematics have had calculus or are studying it concurrently.

Formula 1 can be written as

$$(2) \quad 0 = R(1+i)^n - Si - R. \text{ Let}$$

$$(3) \quad f(i) = R(1+i)^n - Si - R \text{ and solve Formula 2 for } i. \text{ The first derivative of } f(i) \text{ is}$$

$$(4) \quad f'(i) = nR(1+i)^{n-1} - S. \text{ Newton's iteration formula is}$$

$$(5) \quad i_{K+1} = i_K - \frac{f(i_K)}{f'(i_K)}. \text{ Substituting we have}$$

$$(6) \quad i_{K+1} = i_K - \frac{R(1+i_K)^n - Si_K - R}{nR(1+i_K)^{n-1} - S} \text{ for } K = 0, 1, 2, \dots, \text{ which converges to the desired}$$

root under certain circumstances.

For programming the TI83/84, we determine the variables as follows: store  $i_K$  in I,  $R$  in R,  $S$  in S, and  $n$  in N. (Letters on the TI83 are not in italics.)

#### **Example 1**

The following is an example of entering a program and TI 83/84 code that evaluates  $i_{K+1}$  from  $i_K$  and displays  $i_{K+1}$  as I. (See Chapter 16 of the TI83 manual.)

We will first set up an Input program separate from the calculate program. TI83 code and commentary: Press Prgm >> to display the Prgm New menu. Press Enter to select 1: Create New. Then the Name = prompt is displayed and alpha lock is on. Press I N P U T and then Enter to name the program Input. You are now in the program editor. The colon in the first column indicates the beginning of a command line.

Press Prgm > 2 to select 2: Prompt from the I/O menu. Prompt appears on the command line. Press Alpha R , Alpha S , Alpha N , and Alpha I for the first estimate and subsequent estimates for  $i_K, K = 0, 1, 2, \dots$ . Press Enter.

To enter the calculating program: TI83/84 code and commentary: Press Prgm > > to display the Prgm New menu. Press Enter to select : 1Create New. Then the Name = prompt appears, and alpha lock is on. Write C A L C Enter to name the program CALC.

Press Prgm > 3 to select 3:Disp from the Prgm I/O menu. Disp appears on the command line. Press Alpha I Enter to set the program to display  $i_K$ .

Write the following code to enter the expression  $i_K - \frac{f(i_K)}{f'(i_K)} = i_{K+1}$  and store  $i_{K+1}$  in I.

Precede each letter with Alpha. Press I - ( ( R × ( 1 + I ) ^ N - S × I - R ) ÷ ( N × R × ( 1 + I ) ^ ( N - 1 ) - S ) ) → I.

Press 2<sup>nd</sup> Quit for the home screen. Press Clear to clear the home screen. Press Prgm to display the Prgm Exec menu. You may see 1:Input. Press Enter to paste Prgm Input to the cursor. Press Enter to execute the program. You will see R = ?. Write 100 Enter. For S = ?, write 3350.29 Enter. For N = ?, write 30 Enter. For I = ? write .01 Enter for the first estimate  $i_0$ .

To run the program Calc with these inputs, press Prgm 2 Enter for Calc. Press Enter you will see .01 for  $i_0$  and .0080374325 for  $i_1$ .

Again press Prgm 2 Enter for  $i_2 = .00753607$ . Again press Prgm 2 Enter for  $i_3 = .00750011803$ . Again press Prgm 2 Enter for  $i_4 = .00074999963$ . Within the capabilities of the calculator and the program, the convergence to .0075 is complete. (See the Exercises for a word problem and formula with these inputs. See the Exercises for using the TI83/84 TVM Solver for this example.)

This particular program has been written so that the user can participate in the iteration and watch the sequence of estimates converge. It is the task of Calculus to derive Newton's Formula and prove the convergence of the sequence. It is the task of a course in numerical methods to investigate conditions for convergence to selected roots, convergence tests, and more efficient and general programs such as those in the TI83/84 TVM Solver.

### Investigating $f(i)$ (Formula 3) for possible nonnegative roots

Examination of  $f(i) = 0$  reveals a root at  $i = 0$ . Descartes Rule of Signs indicates that there is only one positive root. This information suggest a graphing window of  $X_{min} = -.002, X_{max} = .01, X_{scl} = .001, Y_{min} = -1, Y_{max} = 2, Y_{scl} = 1$ . Trace suggests the root of 0.0075, which we have calculated. The graph indicates a relative minimum at about  $0.0075/2 = 0.00375$ , actually at 0.0038153. To experiment with the results of a

choice of a first estimate, try  $i_0$  as 0.0025 to the left of the relative minimum  $i$  value, and run the two programs. You will see a convergence to the zero root. Later courses will address the topic of the choice of the first estimate. Try  $i_0$  as 0.005 on the other side of the relative minimum  $i$  value. You will see slower convergence to 0.0075. A better understanding of Newton's Formula would predict these results.

**Programming to compute interest rate in the formula for the present value of an ordinary annuity (Fixed Point Method)**

We present the formula in the following notation:

$$(7) \quad A = M \left[ \frac{1 - (1 + R)^{-N}}{R} \right], \text{ where } A \text{ is the present value, } M \text{ is the rent or payment at}$$

the end of each compounding period,  $R$  is the interest rate per compounding period, and  $N$  is the number of compounding periods. Manipulating the formula gives

$$RA = M[1 - (1 + R)^{-N}] \text{ and } (AR - M)(1 + R)^N + M = 0.$$

If we let  $X = 1 + R$  for  $R > 0$ , then  $X^N(AX - A - M) + M = 0$  and

$$(8) \quad 1 + \frac{M}{A}(1 - X^{-N}) = X. \text{ Therefore } X \text{ is a fixed point of}$$

$$(8.1) \quad f(X) = 1 + \frac{M}{A}(1 - X^{-N}) \text{ where } X > 1.$$

For the fixed point method,  $X_{k+1} = f(X_k)$  converges to the desired root of  $f(X) = X$  under certain conditions. The following theorem states more precisely the conditions for this convergence.

**Theorem:** Assume a function  $f$  defined on a closed interval  $J$ , and  $f$  maps  $J$  into itself. If  $f$  is differentiable on  $J$  and  $|f'(X)| \leq k$  for all  $X$  in  $J$  for some nonnegative constant  $k < 1$ , then  $f$  has exactly one fixed point in  $J$ . Furthermore, its value may be determined by first choosing any point  $X_0$  in  $J$ , and then generating the sequence  $X_1, X_2, \dots$  by letting  $X_{k+1} = f(X_k)$  for  $K = 0, 1, 2, \dots$ . This sequence will converge to the fixed point (Conte and deBoor 1980, p. 90).

**Example 2**

To solve numerically Formula 7 and Formula 8 for  $X = 1 + R$  we set up the programs INPUTA and CALCA. To setup INPUTA, press Prgm > > to display the PrgmNew menu. Press Enter to select 1:CreatNew. Then the Name= prompt is displayed and alpha lock is on. Press I N P U T A and then Enter to name the program. You are now in the program editor. Press Prgm > 2 to select 2:Promt from the I/O menu. Promt appears on the command line. Press Alpha A , Alpha M , Alpha N , and X for  $X_0$  and successive estimates which will converge to the desired root. Press Enter.

To setup program CALCA, Prgm > > Enter to get Create New. For Name =, write in C A L C A Enter. Prgm > 3. You see Disp. Write X Enter. Then write the iteration formula  $1 + ( \text{Alpha } M \div \text{Alpha } A ) ( 1 - X \wedge (-) \text{Alpha } N ) \rightarrow X$ . Press 2<sup>nd</sup> Quit and clear the home screen.

To run the programs, Prgm Select PrgmExec. Select InputA. Enter A = 50000, M = 1660.72, N = 36, X = 1.2 for the first estimate. Enter Prgm Select CALCA Enter Enter Enter ... . You see gradual convergence to the desired X = 1.01 for R = .01. (See the Exercises for a problem with this input data and solution.) (This program converges slowly. For an accelerator formula, see Conte and deBoor 1980, p. 98).

### Investigating a present value formula for possible roots

We consider a simpler example that is more manageable on the TI83/84 grapher.

$$(9) \quad 1.833 = 1 \left[ \frac{1 - (1+x)^{-2}}{x} \right], x \neq 0, \text{ and graph } g(x) = -(1+x)^{-2} - 1.833x + 1. \text{ Use the}$$

Window: Xmin=-2, Xmax=2, Xscl=1, Ymin=-10, Ymax=10, Yscl=1. You get an interesting graph of  $g(x)$  with roots 0, 0.06, -1.515, and asymptotes  $x = -1$  and  $g(x) = 1.833x + 1$ , and a relative maximum between 0 and 0.06. Although  $g(x)$  could be considered defined at zero, giving another root of 0, Formula 9 is not. Again, without a careful choice of the first estimate, the program might give unexpected results.

## Exercises

1. Read the theory before Example 1 and do the programming steps and run the programs. Report your device and some of your results. Run program Calc again with a different first estimate such as 0.009. Report the results.
2. Read the theory before Example 2 and do the programming steps and run the programs. Report your device and some of your results. Run program CalcA again with a different first estimate such as 1.1 for  $X$ . Report the results for  $X$  and  $R$ .
3. For the inputs for Formula 1, we give the following word problem. A person was depositing \$100 at the end of each month in a savings account to pay a debt due in 30 months of \$3350.29. What was the monthly and annual interest rate? Substitute into Formula 1 and solve for  $i$ . Report your method of solving and the answer in monthly and annual terms. Try the TVM Solver on the TI83/84.
4. Make up and solve a word problem for the input for the program InputA. Substitute in Formula 7. Report your method of solving and the answer in monthly and annual terms.
5. See if you can squeeze a revision of Programs Input and Calc into an inexpensive scientific calculator with a programming feature. Give the name of the calculator, the price of the calculator, your program code, and the results of the iterations.
6. Use a Newton's Method program to solve for the positive root  $x$  in this word problem. An amount of \$105.97 is saved at the end of each month for 36 months to get \$3938.22. What is  $i$  = the monthly rate? What is the annual rate? Report your experience – point of convergence, first estimate, number of iterations, program code, and calculator. Report any insights you have into Newton's Method. Report a generalization of your method of calculating the first derivative.
7. Use a Fixed Point Method program for solving Formula 9. Use  $X_0 = 1 + x = 1.1$ . Try other values and see what happens.
8. Report your success in graphing Formula 7 with the inputs in program InputA, in the form  $g(R) = -1660.72(1 + R)^{-36} - 50000R + 1660.72$ . Try different windows, Zoom, and Trace. Report your device and what information you manage to get from the effort. See if the TVM Solver can calculate the root.
9. (a) For  $f(x) = (1 + x)^N - 3.4725x - 1$  give an annuity formula. Make a word problem for a practical value of  $N$  and solve it.  
(b) Investigate the graph and roots of  $f(x) = 0$  for  $N = 2, 3, 4, 5$ . What does Descartes Rule of Signs say about roots? What does the TVM Solver report about roots? Estimate the positive and negative roots. Report your results and generalizations.

10. For Formula 9, devise a guess and check method for solving for a positive root. Use bisection – averaging estimates from opposite side of the root to acquire a next estimate. Devise a notation and write out the steps and tests.

11. (a) Solve this two-step word problem. Write the formulas and draw timelines. A person owes \$5000 in 36 months. He has \$1000 to deposit at 2% compounded monthly to pay part of it. How much should he save at the end of each month in the account to pay the rest?

(b) Write and solve your own two-step word problem. See how creative and varied you can be.

(c) Appoint a class committee to check the problems and their answers. Collect the problems for an exercise for the next financial mathematics class.

12. Solve Formulas 1 and 7 explicitly for the variables.

13. (a) Write Formula 7 in terms of  $X = 1 + R$ . For what value of  $X$  is Formula 7 undefined?

(b) Write formula 8.1 for  $M = 1660.72$ ,  $A = 50,000$ ,  $N = 36$ . What is  $f(1.01)$ ?

(c) Graph  $f(X)$  with these given values of  $M$ ,  $A$ , and  $N$ . Use Window: 0, 2, .1, -.5, 1.5, .1.

(d) Press TRACE and track the curve. What does  $f(X)$  approach as  $X$  gets larger? Type 1.01 Enter What do you get?

(e) Use the calculator while in Trace with 2<sup>nd</sup> Calc  $dy/dx$  to calculate the slope of the curve at points close to 0.91, 1, 1.01, 1.02, 1.1. Then calculate the angle of inclination from the  $x$ -axis of the tangent line to the curve in degrees for each point. Record the values  $X$ ,  $Y$ ,  $dy/dx$ , and  $\tan^{-1}(dy/dx)$ .

(f) What is the fixed point for  $f(X)$ ?

(g) Estimate slope at  $X = 0.91$  by

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \text{ with } \Delta x = 0.001.$$

(i) For what value of  $X$  is  $f(X) = 0$ ?

15. Make up a Newton's Method program with InputB and CalcB to solve the present value of an ordinary annuity Formula 7. Follow steps (2), (3), (4), and (5) to set up the iteration formula. The first derivative,  $f'(R) = (-M)(-N)(1 + R)^{-N-1} - A$ . Make up a test problem where all variables are known, with  $0 < R < 1$ , to test the program.

## Side Bar Notes

For many years, unknowns in financial mathematics formulas were found by using as many as four tables that were inadequate for many applications (Bowen, 1976).

Interpolation between table values was often done by hand calculations.

In some of the major financial literature, it is still stated that annuity formulas are solved for interest rate by simple guess and check.

## References

Vest, Floyd, "Computerized Business Calculus Using Calculators, Examples from Mathematics of Finance," *Journal of Computers in Mathematics and Science Teaching*, Vol. 10(4) Summer 1991. (The current article is an adaptation from the 1991 article. The author found it difficult to acquire a pdf copy and permission from the print editor and from electronic resources.)

Conte, S. D. and C. deBoor, *Elementary Numerical Analysis*, Third Edition, McGraw-Hill, 1980.

Bowen, E. K., *Mathematics with Applications in Management and Economics*, Fourth Edition, Richard D. Erwin, 1976.

Texas Instruments, *TI-83 Graphing Calculator Guidebook*, 1996

**Answers to Exercises**

9. (a)  $3.4725 = \frac{(1+i)^N - 1}{i}$ .

(b)

$N$	Descartes Rule	TVM Solver	Roots
2	1 pos. root No neg. roots	1.4725	
3	1 neg. root 1 pos. root	0.15	0.15 and approx. -3.15
4	1 neg. root No pos. roots	-0.09362	
5	2 neg. roots or none No pos. roots	-0.1834	-0.1834 and approx. -2.5

Exercise: Describe the investment results given by the roots for  $N = 2, 3,$  and  $4$ . Can you make the TVM Solver report a root of zero? Graph this  $f(x)$  for  $f(x) = 0$ . What does Descartes Rule of Signs say?

Note: In this formula there is always a zero root. When  $N$  is odd, there is a relative minimum point and a relative maximum point. When  $N$  is even, there is only a relative minimum point. This could be predicted by calculus. This determines the number of roots.

11. (a)  $1000 \left(1 + \frac{0.02}{12}\right)^{36} = \$1061.78$ .  $5000 - 1061.78 = \$3938.22$ .

$$3938.22 = R \left[ \frac{\left(1 + \frac{0.02}{12}\right)^{36} - 1}{\frac{0.02}{12}} \right]. R = \$106.24 \text{ per month saved.}$$

13. (a)  $A = M \left[ \frac{1 - X^{-N}}{X - 1} \right]$  Formula 7 is undefined at  $X = 1$ .

(b)  $f(X) = 1 + \frac{1660.72}{50000} (1 - X^{-36}) = 1 + 0.0332144 (1 - X^{-36})$ .  $f(1.01) = 1.01$ .

(d)  $f(X)$  approaches 1.0332144 as  $X$  continues larger than 1.01.

(e) Some values are:

$X$	0.91	1	1.01	1.02	1.1	1.2
$f(X)$	0.0427559	1	1.01	1.069258	1.0321283	1.0321399
slope	39.18	1.1957	0.827	0.575	0.035	0.0014
$\tan^{-1}(dy/dx)$	$88.53^\circ$	$50^\circ$	$39.6^\circ$	$29.9^\circ$	$2^\circ$	$0.08^\circ$

(In solving for interest rate  $> 0$ , we are limiting the program to  $X > 1$ .)

(f) For  $X > 1$ , the fixed point of  $f(X)$  is at  $X = 1.01$ , since  $f(1.01) = 1.01$ .

(g)  $\frac{\text{rise}}{\text{run}} = 38.3973$ .  $\tan^{-1}(38.3973) = 88.5^\circ$ .

(i)  $f(X) = 0$  at  $X = 0.9081$ .

## Teachers' Notes

For a copy of the manual for the TI84, see <http://www.ti.com/calc>.

If you detect in Exercise 9 that a student uses a reference, have them cite the source. Don't announce this in advance.