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CONSORTIUM

Everybody's Problems

THE Suez Canal

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In the Suez Canal Problem, we consider the real-world, multi-million dollar question of traffic flow through the Suez Canal. The Suez Canal is 193 kilometers long, 300–365 meters wide, and 21 meters deep. The number of ships desiring passage through the canal is essentially the same in both directions in each 24-hour period. The region of the canal available for ship travel is approximately 169 meters in width. This is not enough to allow simultaneous travel in both directions. As a result, the ships must travel in convoys either North-South or South-North.

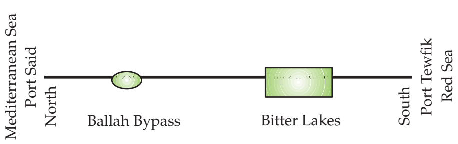

Figure 1

As shown in **Figure 1**, the only positions along the canal that allow for ships to pass are at the Ballah Bypass and the Bitter Lakes. The Ballah Bypass is 10 kilometers in length, stretching from the 50 to 60 kilometer mark south of Port Said. The Bitter Lakes extend for 16 kilometers, from 100 to 116 kilometers south of Port Said. In order for ships to pass, one convoy must pull to the side and dock at floating moorings, since the movement of ships in the opposite direction makes unmoored ships too unstable. Unfortunately, the Ballah Bypass has room for only 17 ships to moor while the Bitter Lakes can accommodate up to 36 ships at a time. Traditionally, the convoy that moves South to North (SN) never stops. The convoys that move North to South (NS) pull aside at the Ballah Bypass and the Bitter Lakes to allow the convoy moving SN to pass. The timing is important so that the convoys will meet each other at the proper places.

Since the SN convoy never docks, its size is not limited by the size of the passing zones. However, the first NS convoy can be at most 36 ships and the second at most 17 ships. This limits the number of ships moving NS to 53 in a 24 hour period. Under the assumption that there must be the same number of ships moving SN as NS, the total shipping each day is therefore limited to 106. But this is much more than can be actually moved through the canal: the size of the canal is not the limiting factor.

Modeling Traffic in the Suez Canal

Just why is it impossible to move 106 ships through the canal? Once we know the answer to that, we can consider how to increase the number of ships that actually are passed through the canal and, in fact, determine the largest number possible. What is needed is a method of representing the movement of a convoy through the canal. We need a mathematical model of traffic through the Suez Canal.

The model we choose for this problem is a simple time-distance graph. **Figure 2** is our model. The vertical axis represents the distance in kilometers along the canal while the horizontal axis represents time measured in hours. The Ballah Bypass and Bitter Lakes are marked. We are simplifying the model and letting the Bitter Lakes extend from kilometer 100 to kilometer 120.

Consider a 10-ship convoy moving NS. Its passage can be described by two lines moving down the page marking the paths of the first and last ships in the convoy. In this example, chosen only to demonstrate the pictorial model in Figure 2, there is a 10 minute separation between ships in the convoy. The first ship in the convoy leaves at 0:00 hours, so the last ship leaves 90 minutes later at 1.5 hours. The convoy travels south at a rate of 15 km/hr, as indicated by the slopes of the lines. The first ship arrives at the end of Ballah Bypass at 4 hours and docks. The last ship in the convoy docks at the north end at 4.83 hours. The convoy waits in Ballah Bypass for 1 hour, then travels south at 20 km/hr to the Bitter Lakes, where it waits for 4 hours. The convoy then completes the trip south at a rate of 10 km/hr. The last ship exits the canal at 17.5 hours. By describing the paths of the first and last ships in each of the three convoys using the graph described above, we can solve the problem of the Suez Canal. Because the 10 ships must maintain a 10 minute separation while they are in transit, the widths of the convoy on the graph is always 1.5 hours, except when they are docked in the Ballah Bypass or the Bitter Lakes.

Standard Ships

In the real Suez Canal, different kinds of ships carrying different cargo can travel at different speeds. In our model, however, we simplify things by considering a "Standard Ship." A Standard Ship travels at 14 km/hr. At this speed, the ships must maintain a minimum separation of 10 minutes. The Egyptian Canal Authority is interested in moving as many Standard Ships through the canal as possible.

Why 106 Ships Will Not Fit in a Day

We are now in a position to see why 106 ships can't pass through the canal each day. The first convoy moving from Port Said to Port Tewfik, called Convoy A, can contain at most 36 ships. If the first ship leaves at 0 hours, then the 36th ship will leave 350 minutes later at 5.83 hours. The last ship in this convoy will arrive at the top of the Bitter Lakes 7.14 hours later, or around 12.97 hours. This is the crossing point for the northbound convoy. The first ship of the northbound convoy (Convoy C) should arrive at the north end of the Bitter Lakes at 12.97 hours. Notice that we ignore stopping time, and the time it takes to dock. Thus, the first ship of Convoy C must leave Port Tewfik at around 8.7 hours to arrive at the north end of the Bitter Lakes at 12.97 hours. If Convoy C is to be as large as possible, it should contain 53 ships. The last ship in the convoy would leave 520 minutes after the first, or at around 17.37 hours. Unfortunately, for the 24 hour schedule to be maintained, this ship must exit from Port Said before 24 hours so that the next Convoy A can have a clear path to the Bitter Lakes. Since it takes 11.43 hours for a complete S-N transit without stopping, the ships in Convoy C cannot clear in time, as shown in **Figure 3**.

Most of the mathematical work on this problem is arithmetic, computing distance, rate, time situations on the graph. Using the graph, we can

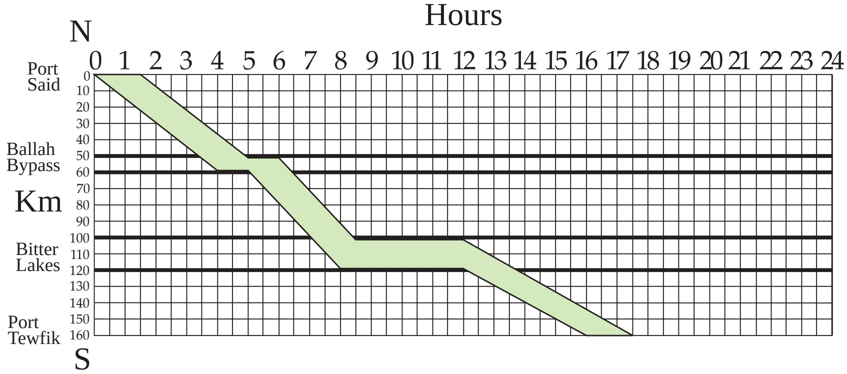

Figure 2: A 10-ship NS convoy moving through the canal.

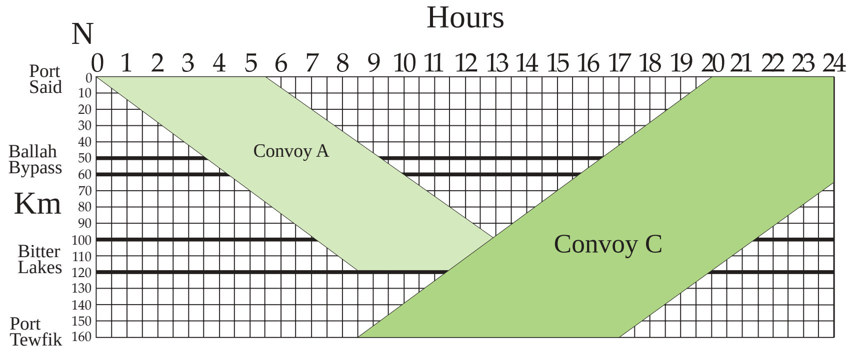

Figure 3: 106 Ships will not fit.

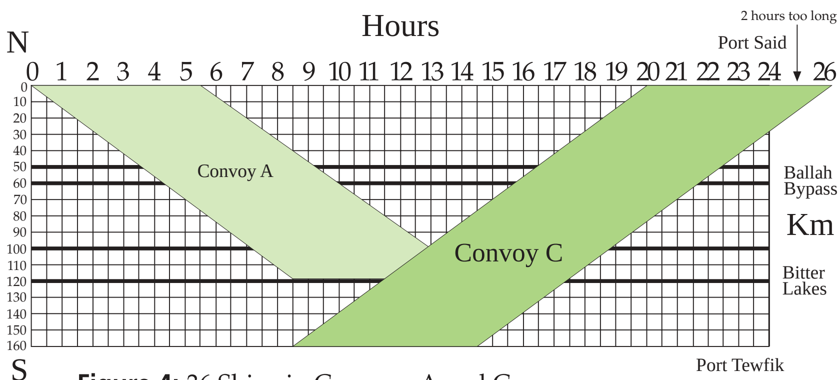

Figure 4: 36 Ships in Convoys A and C.

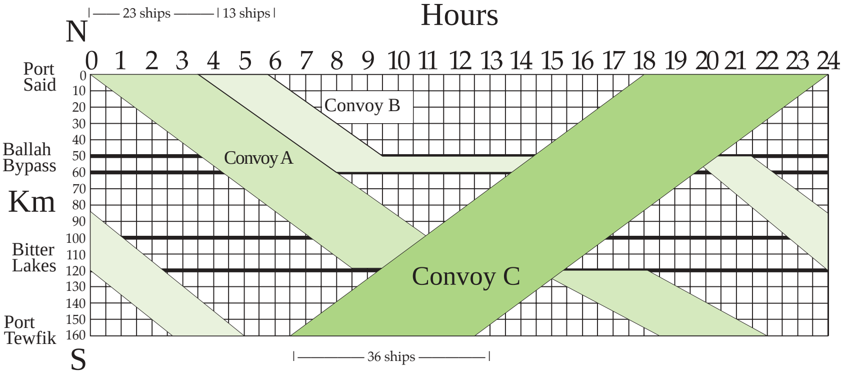

Figure 5: Convoy A with 23, Convoy B with 13, and Convoy C with 36.

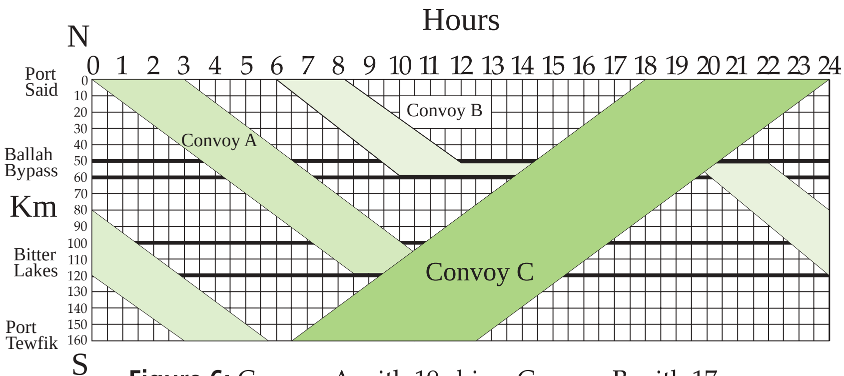

Figure 6: Convoy A with 19 ships, Convoy B with 17 ships, Convoy C with 36 ships

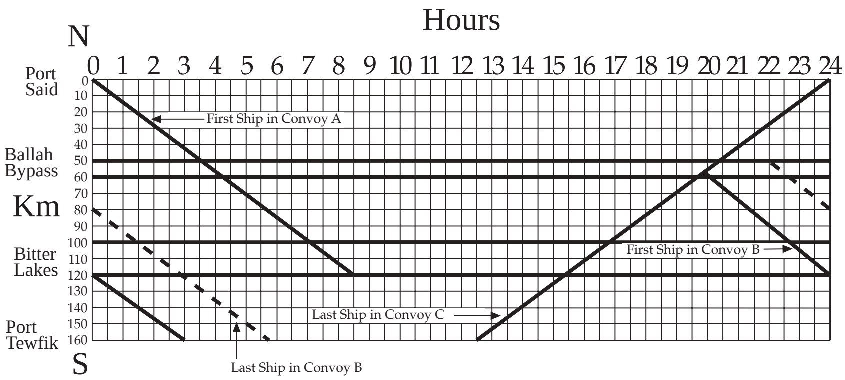

Figure 7: Three Ships That Do Not Change Position

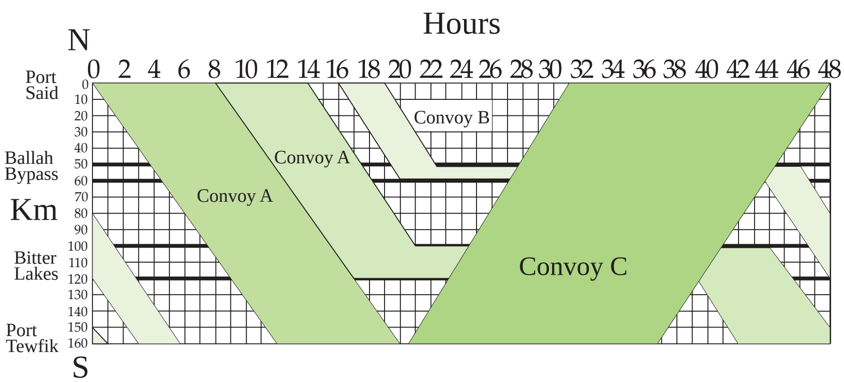

Figure 8: 206 ships in two days

describe pictorially the movement of the ships through the canal. After playing with the system and making sure students understand how the model and how the passing zones work, we attack the fundamental question: What is the largest number of Standard Ships that can pass through the canal in 24 hours?

Place 36 in A and C and Adjust:

A common student approach is to consider the problem of 36 ships in each of Convoys A and C. Students quickly see that this uses too much time. As shown in **Figure 4**, the last ship in Convoy C completes its transit at around 26 hours. This is 2 hours—13 ships—too late. So 13 ships can be taken from Convoy A to give time for all 36 ships in Convoy C. These 13 ships are shifted to Convoy B, giving convoys of size 23, 13, and 36, for a total of 72 ships as shown in **Figure 5**. (Note that Convoy B's inability to complete this trip by hour 24 is not a problem, so long as they complete the transit before the departure of Convoy C the next day, as illustrated by the "wraparound" of Convoy B.)

Other variations on this solution move 15 ships from Convoy A to Convoy B, giving Convoy A with 21, Convoy B with 15, and Convoy C with 36, or using all ships possible in Convoy B by moving 17 ships, giving Convoy A 19, Convoy B 17, and Convoy C 36. All of these result in a total of 72 ships through the canal in a 24 hour period.

Refining the Solution

Students who are careful with the computations and draw accurate graphs realize that the 19, 17, 36 convoy system has some slack in the schedule (see **Figure 6**). The last ship in Convoy A arrives at the 100 km mark at 10.14 hours. The first ship in Convoy C crosses the 100 km mark at

10.7 hours. There is a gap of 0.58 hours, or 4 ships that can be filled by giving two ships to Convoy A and two to Convoy C. Thus, 76 ships can transit the canal with 21 in Convoy A, 17 in Convoy B, and 38 in Convoy C.

All of the solutions presented so far have been based on a trial and error process, moving ships and refining the distribution. Since the solutions were developed in this fashion, students are never sure if they have an optimal solution. How can they show that no solution is better still?

The Algebraic Solution

In all of the graphs presented, there are three ships that do not change their position, regardless of the distribution in the convoys.

(1) The first ship in Convoy A must arrive at the 100 km mark at 7.14 hours.

(2) The last ship in Convoy C must leave at 12.57 hours to arrive at Port Said at 24 hours. It passes the end of the Bitter Lakes at 16.86 hours.

(3) The first ship in Convoy B can leave Port Said at any time, but cannot leave the Ballah Bypass before 19.7 hours. It will complete its transit at 2.86 hours the next day.

If all 17 ships are used in Convoy B, the last ship will complete its transit 160 minutes later, at 5.52 hours. Convoy B will not interfere with Convoy C unless the first ship in Convoy C leaves before 5.52 hours, which would require more than 43 ships in Convoy C, an unreasonably high number. We can use 17 ships in Convoy B. How do you split the ships in Convoys A and C to maximize the traffic flow?

If we let A, B, and C represent the number of ships in Convoy A, Convoy B, and Convoy C respectively, we require that $A + B = C$. If $B = 17$, then $C = A + 17$. We have found one equation in "two-unknowns". If we can develop a second equation in A and C, then we can solve for both. Look again at our pictorial model. Can you see what the second equation is?

Notice in **Figure 7** that no matter what the sizes of Convoy A and Convoy C, the horizontal strip along the northern end of the Bitter Lakes is of constant width. The left endpoint marks the arrival of the first ship in Convoy A and the right endpoint marks the last ship in Convoy C. The distance between the two endpoints is a measure of the number of ship in the two convoys. The time between 7.14 and 16.86 is filled with the combined Convoys A and C. This is 9.72 hours, or 59 ships. So, regardless of how they are divided, we know that $A + C = 59$. Solving the system of equations $A + C = 59$ and $C = A + 17$, we find that $A = 21$ and $B = 38$. The solution found by graphing and adjusting cannot be improved upon given the constraints of the problem.

The 103 Ship Solution

Advanced students working on this problem have found a solution in which you can get 103 ships through per day. The inspiration came from considering faster ships. If the ships moved faster, then the lines drawn on the chart would be steeper and hence more ships would be moved through. But Standard Ships are limited to 14 km/hr. Can we get steeper lines on the chart without changing the speed of the ships?

Consider the graph of $y = x$ on the window $x \in [-10,10]$ and $y \in [-10,10]$ with the graph of $y = x$ on the window $x \in [-20,20]$ and $y \in [-10,10]$. Even though the slope of the line doesn't change, it appears to be more steep in the second graph. The students used this principle to their advantage.

We can make a similar transformation by changing the schedule from a 24 hour to a 48 hour schedule. Using a 48 hour schedule, students can find the solution shown in **Figure 8**. Using a 48 hour schedule, fifty ships in Convoy A' can go straight through from Port Said to Port Tewfik. Thirty-six ships will dock in the Bitter Lakes and 17 in the Ballah Bypass. This allows for a 103 ship convoy to travel north through the canal. Altogether, there are 206 ships traversing the canal in 2 days, or 103 ships per day. ❑

Other Questions for Investigation

1. What would be the maximum number of ships on a 36 hour schedule?
2. If Ballah Bypass were extended to hold 25 ships, how many Standard Ships would be able to transit the canal on a 24 hour schedule?
3. If a new bypass could be built to hold 12 ships, where is the best location on the canal for this bypass?
4. How do the solutions change if the speed is increased to 16 km/hr or decreased to 12 km/hr?

Reference:

Griffiths, J. D, and E. M. Hassan, "Increasing the Shipping Capacity of the Suez Canal", *The Journal of Navigation*, Vol. 31, No. 2, May, 1978.

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Everybody's Problems concerns teaching high school mathematics courses with real-world problems, particularly problems that are suitable for students at all levels.