GEOMETRY & ITS APPLICATIONS

Geometric Probability

Art Johnson

This geometry unit is based on COMAP's HiMAP module 11, Applications of Geometrical Probability, by Fred C. Djang.

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Introduction

The study of probability is a relatively recent development in the history of mathematics. Two French mathematicians, Blaise Pascal (1623–1662) and Pierre de Fermat (1601–1665), founded the mathematics of probability in the middle of the seventeenth century. Their first discoveries involved the probability of games of chance with dice and playing cards. From their original writings on the subject, the study of probability has developed into a modern theory.

Modern probability theory is full of formulas and applications to modern events that are far removed from games of chance. In this unit, you will find the probabilities for situations such as a meteor striking the United States, your meeting a friend at a mall, hearing a favorite song on the radio, and many others.

Instead of using algebraic formulas to solve these probability problems, you will use geometric figures. There are several advantages to using geometry to solve probability problems. The line segments, rectangles, and triangles that are used to solve probability problems are familiar figures. These geometric figures allow you to picture the probabilities of a situation before solving the problem. This will help you develop a sense of a reasonable solution before you solve the problem.

The geometric solutions do not require any memorization of formulas or terms. Instead, you will be able to use well-known geometric relationships to understand the problem situations and then to solve them.

French mathematician Blaise Pascal (1623–1662), co-founder of Probability Theory with Pierre de Fermat, was a deeply religious man who eventually gave up mathematics to devote himself to his religious writings and studies. Yet it was Pascal's friendship with professional gambler Antoine Gombard that led to the founding of Probability Theory. Pascal had met Gombard at several different dinners, at which Gombard had been the center of attention. When Gombard heard Pascal was a mathematician, he asked Pascal for advice about the probability of winning at a dice game. Pascal worked out the mathematics for his acquaintance and gave him the information. Whether Gombard benefited from the information we do not know, but certainly the world of mathematics benefited form Pascal's continued interest in this new field.

BLAISE PASCAL
(1623–1662)

PIERRE DE FERMAT
(1601–1665)

French mathematician Pierre de Fermat (1601–1665) is considered a co-founder of Probability Theory with Blaise Pascal. Fermat was a lawyer and civil servant for the King of France. He was also a family man with children. In spite of all these responsibilities he found time to work with Pascal on Probability Theory, anticipated calculus before it was invented 40 years later, and corresponded with René Descartes about coordinate geometry. He also wrote on many other mathematics topics. How did he find the time? He was a solicitor (government lawyer) and so had to be above the affairs of the local community. Fermat's solution was to stay at home with his family and not engage in the normal social functions of the time. Essentially, Fermat spent all his leisure time at home working at his hobby of mathematics.

SECTION 1

Probability Basics

INVESTIGATION 1.

Put 8 marbles into a bag. Be sure 5 marbles are one color and 3 marbles are a different color. Pick a marble out of the bag. Record the color picked, and return the marble to the bag. Repeat this experiment 40 times. The chance of picking the 3-marble color is 37.5%. This result predicts you would pick the 3-marble color 15 times out of 40. How do your data compare to the prediction? Why do you think they are different? Could they be the same?

The study of probability deals with the **likelihood** of a specific or desired event compared to all the possible events in any given situation. For example, suppose a bag of marbles contains 3 green marbles and 5 red marbles. What is the likelihood or probability of selecting a green marble from the bag? The desired outcome, picking a green marble, would occur 3 times (the number of green marbles) out of 8 possible outcomes (the number of all marbles in the bag). The probability of picking a green marble may be represented as follows:

$$\frac{\text{3 green marbles}}{\text{8 marbles in the bag}} = \frac{3}{8}.$$

The probability of picking a green marble out of the bag is 3/8. The probability of picking a green marble out of the bag may also be represented as a 37.5 percent chance ($\frac{3}{8}$ = 0.375, which converts to 37.5%).

Meteorologists use percentages when they predict the weather. A typical forecast might specify a 40% chance of rain. In this unit we will express all probabilities as fractions, and the percentage equivalent to the fraction will be called a **chance**.

Another way to represent probability problems is with geometric figures. Consider the following problem example:

Sue is throwing darts at the game board in **Figure 1**. What is the chance that she will throw a dart into the shaded area? (Assume that every dart thrown lands somewhere on the game board.)

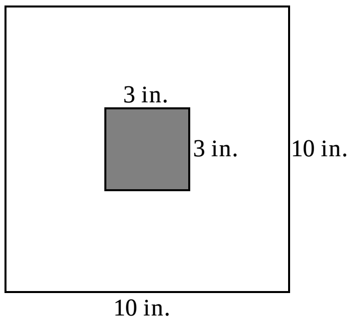


FIGURE 1.

In this case, you compare the areas of the two squares rather than consider the number of darts. The shaded square represents the desired event, or **feasible region**. The entire dart board represents all possible outcomes, or the **sample space**. The probability that Sue will throw a dart in the shaded square may be found by comparing the feasible region to the sample space.

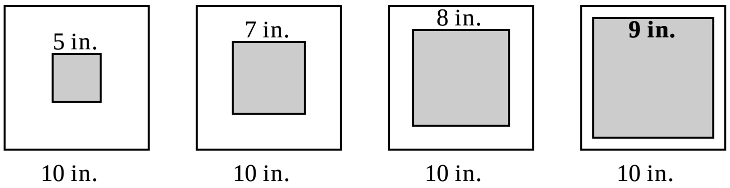
Probability of Sue throwing a dart into the shaded square:
Feasible region = Area of shaded square = 3 in. x 3 in. = 9 sq in.
Sample space = Area of dart board = 10 in. x 10 in. = 100 sq in.

$$\frac{\text{Feasible Region}}{\text{Sample Space}} = \frac{\text{Area of shaded square}}{\text{Area of dart board}} = \frac{9 \text{ sq in.}}{100 \text{ sq in.}} = 0.09.$$

Thus, Sue has a 9% chance of throwing a dart into the shaded square.

What happens to the chance of a dart landing in the shaded square if the shaded square is enlarged? Is Sue more likely to get a dart in a larger shaded square? Notice the diagrams in **Figure 2**.

FIGURE 2.



$$\frac{\text{Feasible region}}{\text{Sample space}} = \frac{25 \text{ sq in.}}{100 \text{ sq in.}} = 0.25 \text{ or } 25\%$$

$$\frac{\text{Feasible region}}{\text{Sample space}} = \frac{49 \text{ sq in.}}{100 \text{ sq in.}} = 0.49 \text{ or } 49\%$$

$$\frac{\text{Feasible region}}{\text{Sample space}} = \frac{64 \text{ sq in.}}{100 \text{ sq in.}} = 0.64 \text{ or } 64\%$$

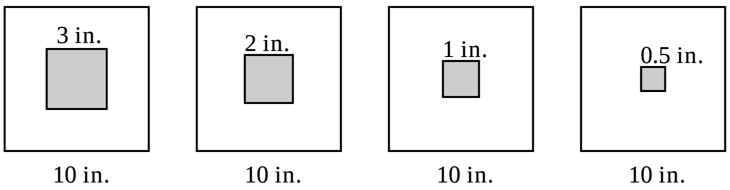
$$\frac{\text{Feasible region}}{\text{Sample space}} = \frac{81 \text{ sq in.}}{100 \text{ sq in.}} = 0.81 \text{ or } 81\%$$

As you can see, the chance that Sue will throw a dart into the shaded square increases as the size of the shaded square increases. If the size of the shaded square continues to increase, Sue's chances of throwing a dart into the shaded square also continue to increase. Suppose that the shaded square increases to ten inches on each side. In that case, every dart that Sue throws will land in the shaded square. Then the probability of a dart landing in the shaded square would be:

$$\frac{\text{Feasible region}}{\text{Sample space}} = \frac{100 \text{ sq in.}}{100 \text{ sq in.}} = 1.00.$$

This means that the chance is 100%, a certainty.

Suppose that the size of the shaded square is reduced. How does that affect Sue's chances of throwing a dart that lands in the shaded square? **Figure 3** shows a series of shaded squares on the same dart board.



$$\frac{\text{Feasible region}}{\text{Sample space}} = \frac{9 \text{ sq in.}}{100 \text{ sq in.}} = 0.09 \text{ or } 9\%$$

$$\frac{\text{Feasible region}}{\text{Sample space}} = \frac{4 \text{ sq in.}}{100 \text{ sq in.}} = 0.04 \text{ or } 4\%$$

$$\frac{\text{Feasible region}}{\text{Sample space}} = \frac{1 \text{ sq in.}}{100 \text{ sq in.}} = 0.01 \text{ or } 1\%$$

$$\frac{\text{Feasible region}}{\text{Sample space}} = \frac{0.25 \text{ sq in.}}{100 \text{ sq in.}} = 0.0025 \text{ or } 0.25\%$$

FIGURE 3.

As you can see with this geometric model, Sue's chances decrease as the size of the shaded square is reduced. If the shaded square is further reduced in size, Sue's chances will decrease even more. What if the square were reduced to the size of a point, with no length or width? In that case, Sue would have no chance of throwing the dart into the shaded square because the tip of the dart would be larger than the point. This means the probability would be:

$$\frac{\text{Feasible region}}{\text{Sample space}} = \frac{0 \text{ sq in.}}{100 \text{ sq in.}} = 0.$$

In terms of percentages, there is a 0% chance.

In this dart board problem, Sue's chances of throwing a dart into the shaded square range from 0% to 100%, depending on the size of the shaded square. The probability of any event happening in any probability problem will always be within this range. In other words, the probability of any event happening is in the range 0 to 1, and the chance of any event happening is in the range 0% to 100%.

An event with a 0% chance is one that will never happen. An event with a 100% chance is one that is certain to happen.

INVESTIGATION 2.

In any problem involving geometric probability, the assumption is that the feasible region is a random part of the sample space. In the dart board problem, for example, the assumption is that the dart is randomly thrown and will land anywhere on the dart board with the same probability. The fact that the feasible region is totally random allows for a prediction based on geometric probability. In some situations, such as dart throwing, the intent is to aim for a particular region, rather than to randomly toss a dart at the entire dart board. It would seem that if there were a conscious effort to throw a dart into a specific part of the dart board, then the actual probability should be higher than the predicted random probability. A simple investigation will show if this is true. Consider a circle inscribed in a square, as shown in **Figure 4** .

Calculate the probability of throwing a penny such that the penny lands completely inside the circle. Assume the throws are random and that every penny lands on the square or the circle. Now draw the inscribed circle onto a sheet of poster board, cut it out, and place it on the classroom floor. Stand back from the poster board and throw pennies into the circle. Record the total number of tosses that land on the target and the number of times the penny lands completely in the circle. Combine your experimental data with your classmates' data. Compare your calculations with the class's experimental data. How do the two probabilities compare? Is your experimental probability higher? Why do you think that is so?

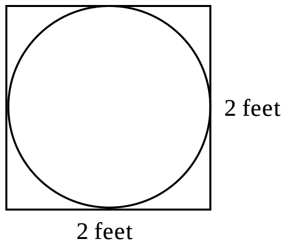


FIGURE 4.

THE MONTE CARLO METHOD

The Monte Carlo approach to problem solving was given its name by physicists working on the Manhattan Project during World War II. The physicists were trying to understand the behavior of neutrons in various materials so that they could construct shields and dampers for nuclear bombs and reactors. Clearly, direct experimentation with nuclear bombs would have been too dangerous, expensive, and time-consuming. So, instead of conducting a direct experiment, the physicists came up with a model that was mathematically the same as the nuclear experiment. Because the model was based on a gambling game, they gave this process the code name "Monte Carlo" after the European city famous for its casinos. Since then, the term "Monte Carlo method" has been applied to the use of chance experiments (either actual or simulated) to estimate values.

Probability problems can be approached in a number of ways. The usual method is through the use of mathematical analysis. However, sometimes the mathematics is too complex or too unrealistic, and an experiment makes more sense. And sometimes the experiments themselves are too costly, dangerous, or otherwise impractical. In such cases, you can use a model that mathematically simulates the original problem. In Investigation 2, for example, you simulated the dart throw by conducting the coin toss experiment. If the simulation involves the use of randomness (like dice, coins, spinners, or a random number generator on a calculator or computer), then you are using a Monte Carlo method. Conducting a simulation a very large number of times will ultimately give you a result that is approximately equal to the result from mathematical analysis.

EXERCISES

1. Give an example of an event with a 0% chance. Give an example of an event with a 100% chance.

2. A spinner for a board game has eight regions of equal size on its face. The regions are numbered consecutively from 1 to 8. Give the probability or chance of each of the following:

   a) spinning a 4

   b) spinning an even number

   c) spinning a multiple of 3

   d) spinning a number greater than 6

   e) spinning a prime number

   f) spinning a number less than 4

   g) spinning a number less than 9

3. The six faces of a cube have been numbered consecutively 1 to 6. Give the probability or chance of each of the following:

   a) rolling a 6

   b) rolling an even number

   c) rolling a prime number

   d) rolling a number less than 3

   e) rolling an odd number

   f) rolling a number greater than 8

EXERCISES

4. Find the probability that a randomly placed point in **Figures 5**, **6**, and **7** is located within the shaded region. Represent the probability as follows:

$$\frac{\text{Measure of feasible region}}{\text{Measure of sample space}} = ?$$

a)

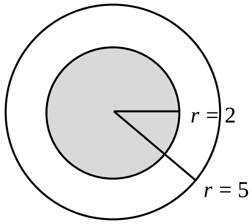


FIGURE 5.

b)

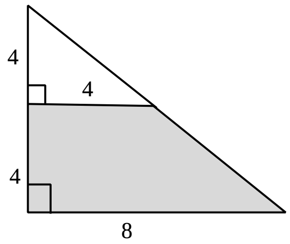


FIGURE 6.

c)

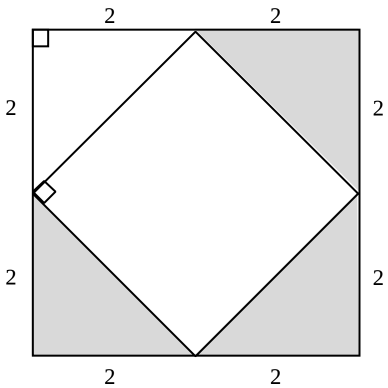


FIGURE 7.

EXERCISES

5. April Showers like to play a game called Contrary Darts. The object of the game is to toss a dart between the squares of the target shown in **Figure 8**, not into the inner square. What is the probability of a random dart landing in the border if every dart hits the target?

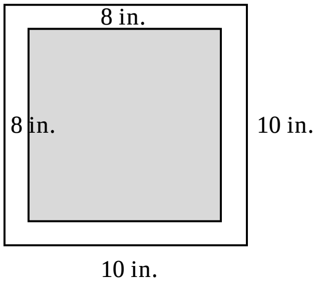


FIGURE 8.

SECTION 2

Linear Models

In Section 1, you learned some basic facts about probability. In the rest of the unit, you will learn how to use various geometric figures to represent a variety of different probability problems. The examples in this section use line segments, the most basic geometric figure, to represent probability. You will notice that although none of the problems in this section involves geometric measure, geometric models can be used to solve them.

EXAMPLE 1

Dwayne Pipe is driving one car in a line of cars, with about 150 feet between successive cars. Each car is 13 feet long. At the next overpass, there is a large icicle. The icicle is about to crash down onto the highway. If the icicle lands on or within 30 feet of the front of a car, it will cause an accident. What is the chance that the icicle will cause an accident?

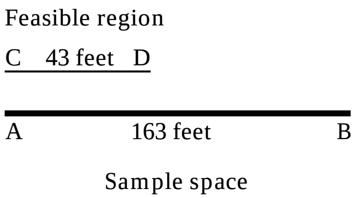


FIGURE 9.

The chance that the icicle will strike a passing automobile may be represented by line segments. The sample space is the distance from the front of one car to the front of the next car. The feasible region is the 30 feet given in the problem plus the length of the car (13 feet). Both of these may be represented as segments (**Figure 9**).

Segment AB represents the sample space.
Sample space = 150 feet (distance between cars) + 13 feet (length of a car) = 163 feet.

The feasible region may be represented by segment CD.
Feasible region = 30 feet (danger zone) + 13 feet (length of a car) = 43 feet.
The probability of the icicle causing an accident may be found by comparing the feasible region to the sample space.

$$\frac{\text{Measure of feasible region}}{\text{Measure of sample space}} = \frac{\text{Length of segment CD}}{\text{Length of segment AB}} = \frac{43 \text{ feet}}{163 \text{ feet}} = 0.264.$$

The chance that the icicle will cause an accident is less than 30%.

EXAMPLE 2.

Mandy Torpedoes sells 43 different flavors of ice cream. In addition to selling ice cream cones and sundaes, Mandy also fills quart containers with ice cream for take-out. What is the probability that Mandy will have to open a new five-gallon ice cream container to fill Art Deco's order of a quart of Cookie Dough ice cream?

To solve this problem, represent the feasible region and the sample space with line segments. The sample space of five gallons may be represented by segment EF, as shown in **Figure 10**. Each gallon unit on segment EF has been divided into four equal units to represent quarts.

The feasible region of one quart may be represented by the last one-quart interval. If Q is in the last one-quart interval of the five-gallon container EF, then Mandy will have to open a new container. If Q is located anywhere else on EF, then Mandy will not have to open a new container. In other words, the feasible region is when Mandy has to open a new five-gallon container. This happens only if Art's order comes when the five-gallon container is nearly empty, or when Q is in the last one-quart interval.

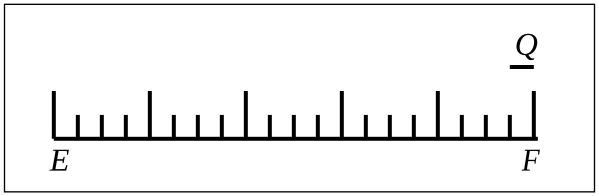


FIGURE 10.

The probability that Mandy needs to open a new five-gallon container of Cookie Dough ice cream to fill Art Deco's order is found by comparing the feasible region to the sample space.

$$\frac{\text{Feasible region}}{\text{Sample space}} = \frac{\text{Length Q}}{\text{Length EF}} = \frac{1 \text{ quart}}{20 \text{ quarts}} = 0.05 \text{ or a 5\% chance.}$$

As in Example 1, the solution to this problem was found by comparing the feasible region to the sample space. Although neither of the problems involved geometric measures or geometric figures, line segments could be used to represent the ideas.

INVESTIGATION 3.

Videotape a 30-minute network television program. Without previewing the tape, fast-forward through the program with the television turned off, and then stop the tape. Turn on the television, play the tape starting at the point where you stopped it, and determine whether the program or a commercial is being broadcast. Record your results on paper. Turn off the television, rewind the tape to the beginning again, and fast-forward to another random place in the tape. Repeat the procedure ten times. From your data make a prediction about the number of minutes commercials are aired during this 30-minute program. Now watch the program and time the total length of the commercials to check the accuracy of your prediction.

BUFFON'S NEEDLE EXPERIMENT

Georges Louis LeClerc, the Compte de Buffon (1707–1788), was a mathematician who is now famous for his needle experiment involving probability. In this experiment, a needle of length L is randomly dropped onto a plane of evenly spaced parallel lines (d is the space between the lines). Using mathematical analysis, Buffon showed that when the length of the needle is the same as the distance between the parallel lines, $L = d$, the probability that the tossed needle would land crossing one of the lines is $\frac{2}{\pi}$. So, if H is the number of hits (the needle lands crossing a line) and T is the number of tosses, $L = d$, $\frac{H}{T} = \frac{2}{\pi}$. He further calculated that if the length of the needle is the same as or less than the space between the parallel lines, the probability p that the needle lands crossing a line can be described as $p = \frac{2L}{\pi d}$.

Buffon then actually performed the original experiment ($L = d$), repeatedly tossing a needle onto a plane of parallel lines. By putting his actual values into his equation $\frac{H}{T} = \frac{2}{\pi}$ and solving for π, Buffon got an experimental value for π that approximates the calculated value of π (3.141592).

You did the same type of thing in Investigation 2 on page 11 in which you compared your experimental results to your mathematical analysis of the probability that a tossed penny would land completely within a defined area. Recall from the discussion of the Monte Carlo method on page 12 that the experiment needs to be repeated many times in order to approximate the mathematically calculated value. Contemporary mathematicians have used computers to simulate Buffon's needle experiment, repeating the process 100,000 times in order to get an experimental value of π that approximates the calculated value.

EXERCISES

1. Holly Mackerel and Patty Cake are driving from New York City to Washington, D.C., a distance of about 300 miles. Their car has a broken gas gauge, but Holly knows her car's gas tank holds exactly enough gas to make the trip without having to stop for gas. Unfortunately, they hit bad weather, which causes traffic delays, and they run out of gas. What is the probability that they will be within 50 miles of Washington when they run out of gas?

2. Tommy Gunn is in charge of keeping the water cooler filled for his football team, the Bali High Bobcats. The cooler holds 7 gallons, and it should have at least 5 gallons in it at the end of team practice, when the whole team will be drinking water. Tommy refills the cooler whenever it contains only one gallon of water. Tommy does not check the cooler at the end of practice. What is the probability that the cooler will hold at least 5 gallons of water at the end of practice?

3. Allison Trouble works for the telephone company. Her job is to help repair downed telephone lines. The company sends her on a repair call for a downed wire on Frankie Lane. The telephone poles on Frankie Lane are 50 feet apart. If the break is within 5 feet of a pole, Allison can connect the ends of the wire from her ladder. If the break is not within 5 feet of one of the telephone poles, then Allison will have to call a repair truck to reattach the wires.

EXERCISES

a) What is the probability that Allison can reattach the wire herself?

b) Allison's company has found that there is an average of 20 line breaks in a typical day. A line crew with a truck can repair two breaks per day. How many trucks and line crews should the company maintain?

4. Matty Door listens to Slim Picken's Top Ten Countdown Show every Saturday on the radio to hear the Top Ten songs. This week Matty knows his favorite group, Joe Elastic and His Rubber Band, will have their latest song, "Sappy Music," in the Top Ten. Just as the show begins, Matty's mother asks him to run an errand. He finishes the errand when the countdown is about to reach song number 4. What is the probability that Matty will get to hear the new song by Joe Elastic and His Rubber Band?

5. Jim Dandy rides a bus to school every day. On the way to school, Jim's bus passes through a traffic light. The light cycle of the traffic light is 20 seconds red, 5 seconds yellow, and 50 seconds green. What is the probability that Jim's bus will get a green light at the traffic light?

EXERCISES

6. Minnie Soda drinks a 4-oz. glass of diet cola every day. Her father, mother, and brother also drink diet cola every day. What is the probability that, on a given day, the half-gallon of diet cola in her refrigerator will not contain enough cola for her full 4-oz. glass?

7. Belle Tower brought her camera film from a recent trip to Washington, D.C., to the Someday My Prints Will Come Photography Shop. Unfortunately, the shop ruined 4 photos in a row from Belle's 24-exposure roll. What is the probability that the ruined photos included the eighth, ninth, or tenth photos on the roll (the photos of the White House)? (Hint: Think about the possible locations of the first of the ruined photos.)

8. Otto Mation has bought a copy of the new tape "Face Burn" by Cement. The A side of the tape is 30 minutes long and contains his favorite song "Homework Blues." The song "Homework Blues" lasts 4 minutes. Alas, Otto's sister uses the A side of the tape to record her favorite song "Lunch Time Laughs" at the beginning of the tape. "Lunch Time Laughs" lasts 8 minutes. What is the probability that all of Otto's favorite song is still on side A of the tape? (Hint: What is the feasible region for the start of "Homework Blues"?)

SECTION 3

Area Models

As you saw in Section 1, some probability problems may be solved with area models rather than line segments. The feasible region and the sample space in the problems in this section are represented by circles, rectangles, triangles, and trapezoids.

EXAMPLE 1

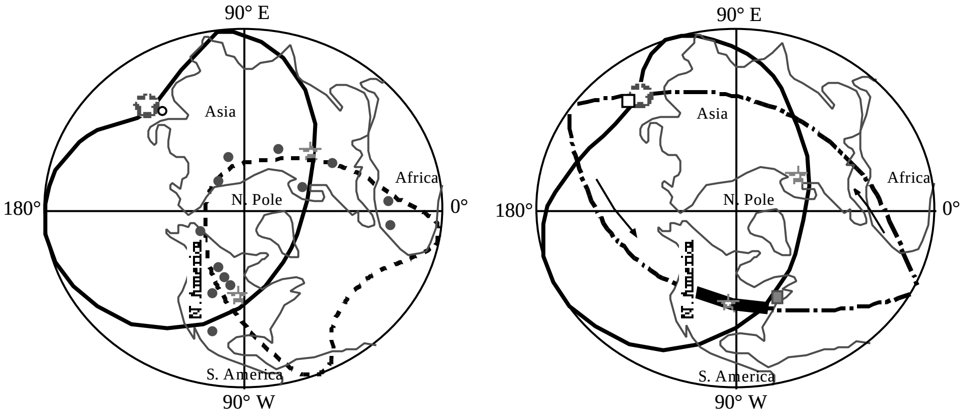
AN AERIAL VIEW OF THE BARRINGER METEOR CRATER, ARIZONA, WHICH HAS PLAYED MAJOR ROLES BOTH IN CRATER RESEARCH AND AS A TRAINING SITE FOR EACH OF THE APOLLO ASTRONAUT CREWS. PHOTO BY D. J. RODDY AND K. A. ZELLER, U. S. GEOLOGICAL SURVEY, FLAGSTAFF, ARIZONA.

A current scientific theory about the fate of dinosaurs on the earth suggests that they were made extinct by the effects of a large meteor which struck the earth in the Caribbean Sea, off the coast of Central America. The United States has been struck by large meteors in the past, as shown by the crater in Arizona at left. What is the chance that a meteor striking the earth will land in the United States?

The probability relationships for this problem may be represented by areas. The sample space is the surface area of the earth, or 196,940,400 square miles. The feasible region is the area of the United States, or 3,679,245 square miles. As with the linear model problems, the probability of a meteor hitting the United States may be found by comparing the feasible region to the sample space:

$$\frac{\text{Feasible region}}{\text{Sample space}} = \frac{\text{Area of United States}}{\text{Area of the earth}} = \frac{3{,}679{,}245}{196{,}940{,}400} = 0.019.$$

The chance that a meteor hitting the earth will land in the United States is about 2%.



- Crater, 50 million to 100 million years old, N. Hemisphere
- Crater, 50 million to 100 million years old, S. Hemisphere
- K-T crater, 65 million years old, N. Hemisphere
- A circle connecting nodes from the Phanerozoic era
- K-T swath connecting craters 50 million to 100 million years old.

- Cratering node, N. Hemisphere
- Cratering node, S. Hemisphere
- Crater, late Devonian, N. Hemisphere
- Crater, late Devonian, S. Hemisphere
- Curve connecting numerous craters of the late Devonian period, roughly 360 million years ago
- Inferred path of fireballs in 1913 observed from Canada to Bermuda

RANDOM METEORS

Example 1 in this section asked about the probability of a meteor landing in the United States, assuming that meteors strike the earth randomly. Until recently, it was indeed believed that meteors struck the earth randomly. However, geophysicist Herbert R. Shaw has gathered evidence that suggests meteors strike the earth in patterns, not randomly. Shaw hypothesizes that unevenly distributed matter inside the earth produces an uneven gravitational influence on meteors. This pulls meteors into particular orbits leading to specific impact patterns.

The diagrams above show the patterns in which meteors have fallen on the earth. Note that these patterns would be more circular around a three-dimensional globe than they appear in these diagrams. You can see that each of these paths passes through the United States. So, assuming that Shaw's hypothesis is correct, if a meteor falls, the United States in still in danger of being hit.

EXAMPLE 2

Patty Cake is planning a trip to Budapest, the capital of Hungary, where she plans to visit the homes and gravesites of her ancestors. The city of Budapest is divided by the Danube River into two parts, Buda and Pest. Although Patty will spend most of her trip in the Buda portion of Budapest, she is willing to stay at the hotel with the best room value for the money and to pay travel costs to the places in Budapest she wants to see. Patty told her travel agent to reserve a room in any one of the top-value hotels in Budapest. What is the probability that Patty will stay in a hotel room located in Buda?

The solution to Patty's problem may be represented in an area model. The probability is found by comparing the feasible region to the sample space.

The diagram in **Figure 11** may be used to represent the approximate dimensions of the city of Budapest.

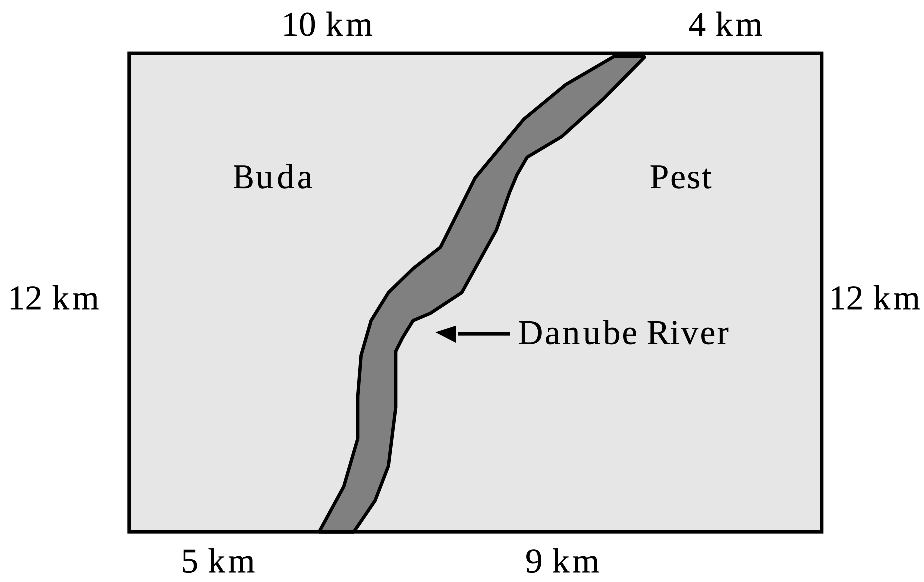


FIGURE 11.

This problem assumes that the number of hotels is proportional to the area of the city and that the hotels are evenly distributed throughout the city. Thus, Buda is the feasible region, and the entire city of Budapest is the sample space.

$$\text{Probability} = \frac{\text{Feasible region}}{\text{Sample space}} = \frac{\text{Area of Buda}}{\text{Area of Budapest}}.$$

The area of Buda may be found by approximating its shape as a trapezoid.

$A = \frac{1}{2}(b_1 + b_2)h = \frac{1}{2}(5 + 10)(12) = 90$ sq km.

Area of Buda = 90 sq km.

The area of the entire city of Budapest may be found by finding the area of the rectangle that approximates the area of the city.

$A = bh = (5 + 9)(12) = (14)(12) = 168$ sq km.

Area of Budapest = 168 sq km.

$$\frac{\text{Feasible region}}{\text{Sample space}} = \frac{\text{Area of Buda}}{\text{Area of Budapest}} = \frac{90 \text{ km}}{168 \text{ km}} = 0.536.$$

The chance that Patty will stay in a hotel in Buda is about 54%

INVESTIGATION 4.

In recent years, tennis rackets have been made of graphite instead of wood. Such advances have made possible larger heads for tennis rackets. A larger head should mean an increased probability that a player will hit the ball. Check out an old tennis racket and a new tennis racket and compare the probability of hitting a ball with a new one and an old one. Test your findings by hitting a tennis ball against a wall with an old wooden racket and then with a new larger racket. If the probability increases with a larger racket, why don't manufacturers produce even larger heads on tennis rackets?

EXAMPLE 3

Moe Mentum owns a goat that will eat anything, especially the tennis balls that are hit into Moe's yard from a neighboring tennis court. To keep his goat from eating more tennis balls, Moe decides to tether it in a corner of his yard at Point A, as shown in **Figure 12**. The tether is 20 feet long. What is the probability that a tennis ball hit into Moe's yard will be within reach of the tethered goat?

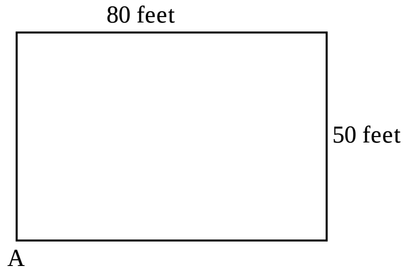


FIGURE 12.

To find the probability, compare the feasible region to the sample space. The feasible region is the area that the goat can reach without breaking its 20-foot-long tether. The measure of the sample space is the area of Moe's yard.

The feasible region is represented in **Figure 13**.

The feasible region, the area that the goat can reach, is part of a circle. Since the angle at Point A is 90°, the feasible region is $\frac{90°}{360°}$, or $\frac{1}{4}$ of a circle with a radius of 20 feet. The probability may be represented as follows:

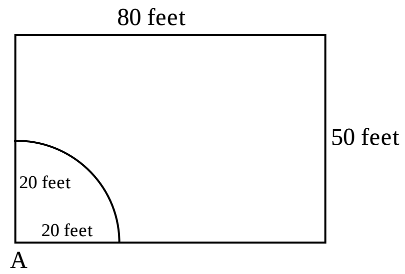


FIGURE 13.

Feasible region = Area of the goat's reach = $\frac{1}{4}\pi r^2$
$= \frac{1}{4}\pi(20)^2 = 100\pi = 314$ sq ft.
Sample space = Area of Moe's yard = $b \times h = 80 \times 50 = 4000$ sq ft.

The probability that Moe's goat will eat a tennis ball may be found by comparing the feasible region to the sample space.

$$\frac{\text{Feasible region}}{\text{Sample space}} = \frac{\text{Area of goat's reach}}{\text{Area of Moe's yard}} = \frac{314 \text{ sq ft}}{4000 \text{ sq ft}} = 0.079.$$

The chance of Moe's goat eating a tennis ball hit into his yard is about 8%.

EXERCISES

1. Washington, D.C. was originally laid out as a square with sides ten miles in length. On a visit to Washington, Miles Away plans to visit all the important sites, such as the White House, the Smithsonian Museum, the Capitol, and the National Zoo. What is the probability that any one of these sites is within a mile of the center of Washington?

2. Cher Cropper and Luke Warm are designing a dart board for their school fair. The diagrams in **Figure 14** show the pattern each student prefers for the dart board design. Which pattern has the greater probability for a contestant to throw a dart into the shaded area?

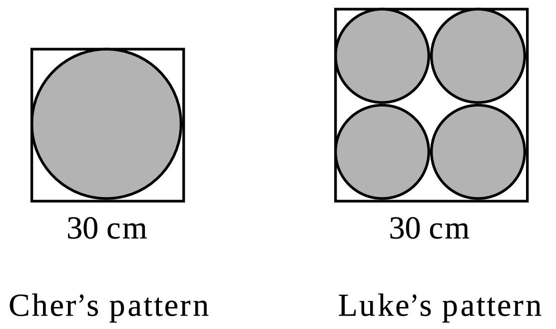


FIGURE 14.

EXERCISES

3. Sally Forth is practicing her golf game at the school athletic field. She accidentally hits a shot toward the street. As her ball bounces down the sidewalk, it hits a sewer grating. What is the probability that the ball will drop right through the grating instead of hitting one of the grates and bouncing away? The diameter of her golf ball is 4 cm. The sewer grating is made up of squares, each 10 cm on a side. Assume that the thickness of the grating is negligible and that a ball falls through only if it completely clears the grating. (Hint: Think about the feasible region for the center of the ball.)

4. Moe Mentum's goat is at it again. This time Moe decided to tether his goat midway along the longer side of his yard. The tether is still 20 feet long.

   a) What is the probability that any tennis balls which are hit into Moe's yard will be within the goat's reach?

   b) What is the probability of the goat eating the ball if Moe shortens the length of the goat's tether to only 10 feet?

5. Lauren Order is trying her skill at a shooting gallery in an amusement park. A series of tangent circles, each 1 inch in diameter, is hung on the wall of the shooting gallery. Lauren will win a huge stuffed panda if the bullet she fires is completely within any of the circles. The bullet makes a hole with a diameter of 1/4 inch. Lauren is a good enough shot that the center of her bullet will certainly be in one of the circles. What is the probability that Lauren will win the panda?

EXERCISES

6. The game of shuffleboard was introduced in Daytona Beach, Florida, in 1913. In shuffleboard, a disk 6 inches in diameter is pushed by shuffleboard poles on the court shown in **Figure 15**. Points are scored by getting a disk completely within one of the scoring areas. For example, if a disk lands completely within 10 OFF, then 10 points are deducted from the score. What is the probability that a disk which clears the dead lines will land in one of the 7-point areas?

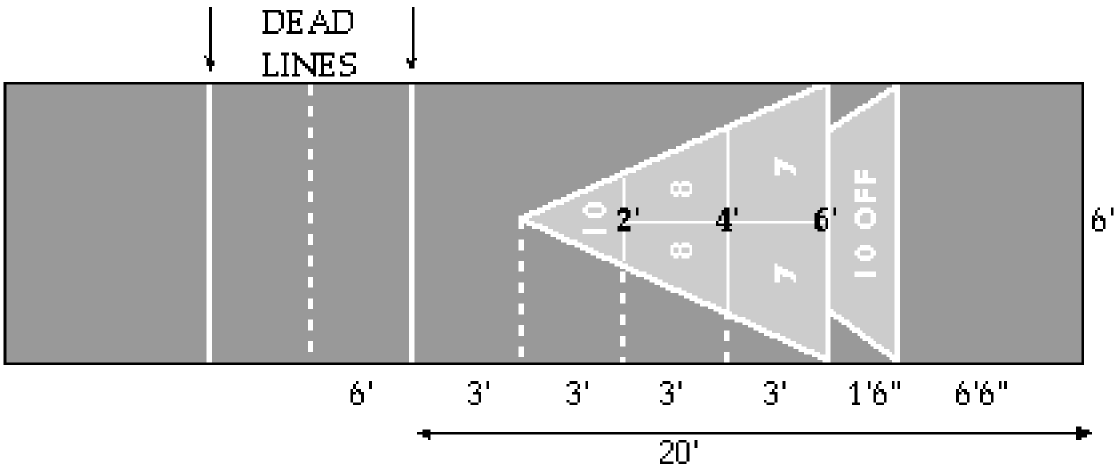


FIGURE 15.

EXERCISES

7. Wee Willie Keeler, an outstanding baseball player during the last part of the nineteenth century, explained his success in professional baseball very simply: "Hit 'em where they ain't." Keeler might have been able to use geometric probability to predict whether a batted ball would land in an opponent's baseball glove.

   Assume you are batting in a game at Chez Vous Stadium, a new stadium with fences a uniform distance of 360 ft. from home plate. The three outfielders can each cover a different amount of the outfield in the 4 seconds it takes a hit ball to travel to the outfield. The left fielder, Harry Legs, can cover a circle with radius 90 ft. The center fielder, Holly Mackerel, can cover a circle with radius 75 ft. The right fielder, Dusty Rhodes, can cover an area with radius 80 ft.

   a) What is the probability that a pop fly will hit the ground (and not an outfielder's glove) in the 4 seconds it takes for a pop fly to come down to earth? (Assume the fielders are placed in such a way that their areas do not overlap, and that infielders will catch any pop fly hit less than 160 ft. from home plate.)

   b) If the fielders' areas do overlap, is the probability of hitting safely greater or less?

SECTION 4

Coordinate Geometry Models

Another model we can use to represent probability problems is based on geometric figures placed on a coordinate grid. The problems in this section demonstrate how a coordinate system may be used to solve probability problems.

EXAMPLE 1

Gail Force is operating a small air-freight delivery service. Every day she travels from her home base to another city and back. The cities she flies to are all less than 200 miles from her home base. Every morning Gail puts enough fuel into her small plane to fly 450 miles.

This morning, Gail is distracted by a minor repair to her plane and forgets to refuel. What is the probability that she will run out of fuel during her flight today?

This problem may solved by using a coordinate system and comparing the feasible region with the sample space. The problem is represented on the coordinate system in **Figure 16**.

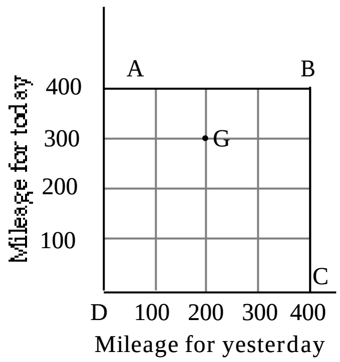


FIGURE 16.

In this coordinate display, the horizontal axis, or *x*-axis, represents the mileage flown yesterday. The vertical axis, or *y*-axis, represents the mileage flown today. Since the longest round-trip flight is 200 + 200 = 400 miles, the mileage traveled on either day may be represented within the square shown on the graph. Any point on the square ABCD or in its interior represents one possible case for Gail Force's travel during these two days. For example, Point G (200, 300) represents an itinerary that took Gail on a 200-mile round trip yesterday and a 300-mile round trip today. Square ABCD represents the sample space.

The total mileage possible on a full tank is 450 miles. The total mileage for yesterday and today combined must not exceed 450 miles. If Gail's mileage does exceed 450 miles, she will run out of fuel. This is represented on the graph by the area $x + y > 450$. (See **Figure 17**.)

Since Point Q has a y-coordinate of 400 and is also on line $x + y = 450$, the x-coordinate of Point Q must be 50. The distance from Point Q (50, 400) to Point B (400, 400) is 350. BQ has a length of 350.

Since Point T has an x-coordinate of 400 and is also on line $x + y = 450$, then the y-coordinate of Point T must be 50. The distance from Point T (400, 50) to Point B (400, 400) is 350. BT has a length of 350.

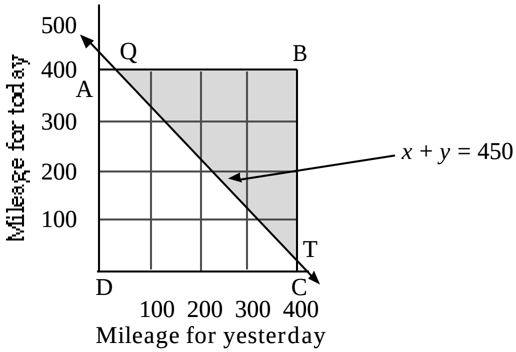


FIGURE 17.

Triangle QBT represents the cases during which Gail will run out of fuel, the feasible region. The probability that Gail will run out of fuel is found by comparing the feasible region to the sample space.

Area of triangle QBT $= \frac{1}{2}bh = \frac{1}{2}$(350 miles)(350 miles) = 61,250 sq mi.
Area of square ABCD $= bh$ = (400 miles)(400 miles) = 160,000 sq mi.

$$\frac{\text{Feasible region}}{\text{Sample space}} = \frac{\text{Area of triangle QBT}}{\text{Area of square ABCD}} = \frac{61{,}250 \text{ sq mi}}{160{,}000 \text{ sq mi}} = 0.383.$$

The chance that Gail will run out of fuel today if she forgot to refuel is about 38%.

RENÉ DESCARTES

French mathematician René Descartes (1591–1650) found the inspiration for coordinate geometry one morning while lying in bed. Descartes had developed the habit of lying in bed and thinking until he felt like getting up. Descartes was such a good student that his instructors made this concession to his genius. He continued this routine into adulthood. During one of these morning sessions, he noticed a fly crawling across the ceiling. As he watched the fly, Descartes began to think of how he could describe the fly's path without tracing it, that is, how to describe the path mathematically. The result was his system for coordinate geometry, which we use today.

EXAMPLE 2

Two friends, Candy Cane and Ann Archy are shopping at the mall. They agree to split up for a time and then meet for lunch. They plan to meet in front of Harold Square Clothing Store between 12:00 noon and 1:00 PM. The one who arrives first agrees to wait 15 minutes for the other to arrive. After 15 minutes, that person will leave and continue shopping. What is the probability that they will meet if each of the girls arrives at any time between 12:00 noon and 1:00 PM?

This probability problem may be represented on a coordinate system. Since either of the two girls may arrive at the clothing store at any time within the entire hour, the sample space may be represented as shown on the coordinate axes in **Figure 18**. Any point within the rectangle may represent the arrival time of the two girls. Point Z (50, 20) represents the arrival of Candy after 50 minutes (12:50) and the arrival of Ann after 20 minutes (12:20). All other arrival possibilities may be represented by points in rectangle QRST or on its boundaries. Thus, the sample space is the rectangle QRST, which has an area of 30 x 60, or 3600 units.

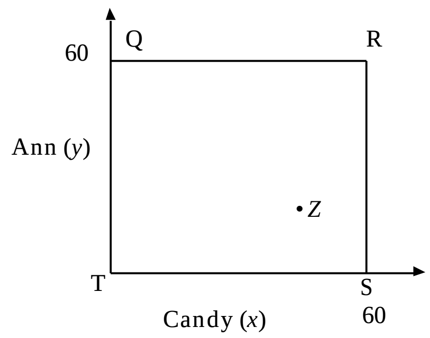


FIGURE 18.

The feasible region is found by representing the 15-minute waiting period. It is actually made up of two regions: one representing the case when Ann arrives first, and the other representing the case when Candy arrives first. The two regions lie on either side of the line $x = y$, which represents the case in which the two girls arrive simultaneously (**Figure 19**).

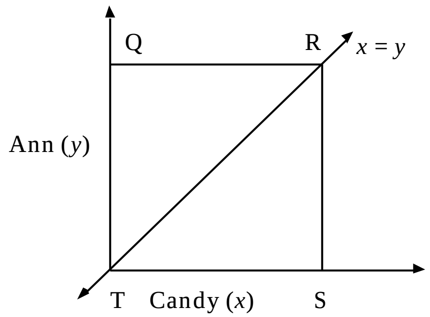


FIGURE 19.

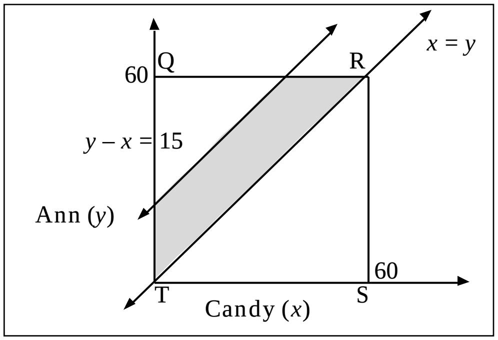


FIGURE 20.

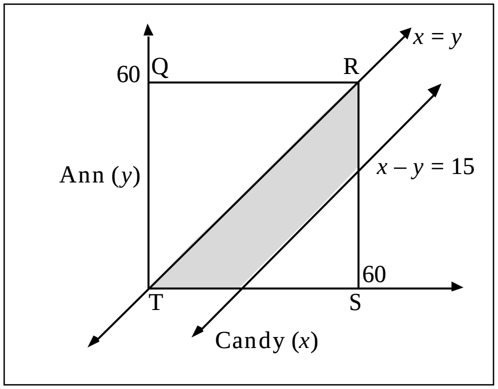


FIGURE 21.

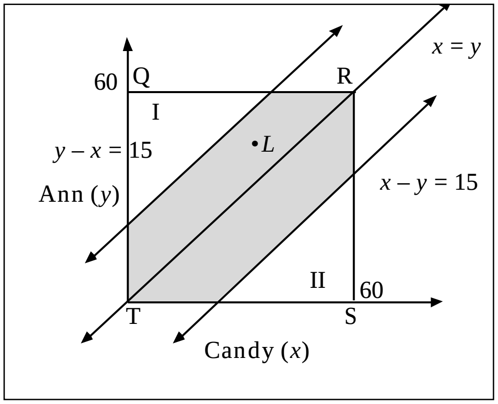


FIGURE 22.

If Candy arrives first, then Candy (x) and Ann (y) will meet if $y - x < 15$.
If Ann arrives first, then Ann (y) and Candy (x) will meet if $x - y < 15$
These two inequalities may be represented on the coordinate axes as shown in **Figures 20** and **21**.

The two inequalities are combined in **Figure 22** to show the feasible region.

The union of the two shaded areas is the feasible region. If the arrival times of the two girls can be represented by a point within either shaded area, then they will meet. For example, Point L (30, 42) is within the feasible region. Since it is within the feasible region, it represents a set of arrival times for the two girls so that they will meet. If Candy arrives at 12:30 and waits 15 minutes, she will be waiting when Gail arrives at 12:42.

The area of the feasible region is equal to the sample space less the areas of the two triangles, labeled triangle I and triangle II. (Note that both triangles are isosceles with sides of 45 units.)

Feasible region = area of rectangle QRST – area of 2 triangles.
Feasible region = $3600 - 2(\frac{1}{2})(45)(45) = 1575$ sq units.
The probability that the two girls will meet may be found by comparing the feasible region to the sample space.

$$\frac{\text{Feasible region}}{\text{Sample space}} = \frac{1575 \text{ units}}{3600 \text{ units}} = 0.438.$$

The chance that the two girls will meet between 12:00 noon and 1:00 PM if each one waits 15 minutes for the other to arrive is about 44%.

EXAMPLE 3

A line segment is 8 inches long. Two points are put on the segment at random locations. What is the probability that the three segments formed by the two points will make a triangle?

The first step is to determine the relationship among the three segments. The problem may be pictured as shown in **Figure 23**.

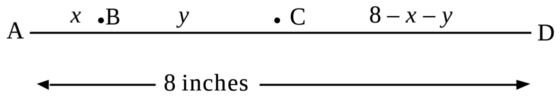


FIGURE 23.

What is the probability that segment AB, segment BC, and segment CD will make a triangle?

This problem may be solved by using a coordinate system.
Let the length of segment AB = x.
Let the length of segment BC = y.
The length of segment CD = $8 - x - y$.
The length of AB must be less than 8 inches, and the length of BC must be less than 8 inches. This may be represented on a coordinate system as shown in **Figure 24**.

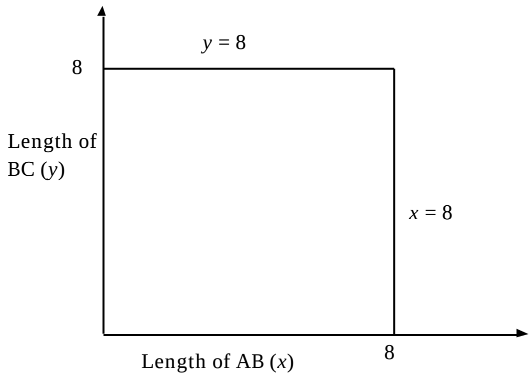


FIGURE 24.

In addition, the sum of the two segments also must be less than the total length of the original segment, so AB + BC is less than 8 inches. This may be written $x + y < 8$. This inequality is shown on the coordinate system in **Figure 25**.

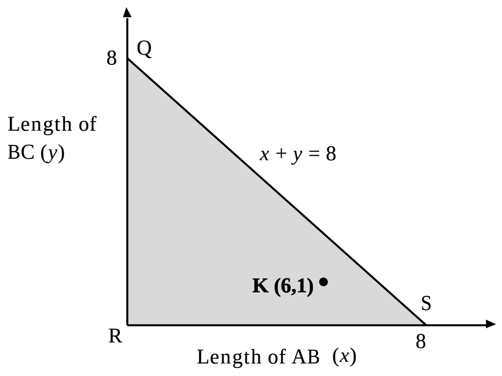


FIGURE 25.

The area of triangle QRS is the sample space. All the points in triangle QRS represent possible lengths for segment AB and segment CD. Point K, for example, has coordinates (6, 1). Point K represents a length of 6 for segment AB, and a length of 1 for segment BC.

The relationship between the three sides of a triangle is as follows: The sum of the lengths of any two sides of a triangle must be longer than the length of the third side. This results in the following inequalities, all of which must be true:

1. $x + y > 8 - x - y$ (AB + BC > CD)
2. $x + (8 - x - y) > y$ (AB + CD > BC)
3. $y + (8 - x - y) > x$ (BC + CD > AB)

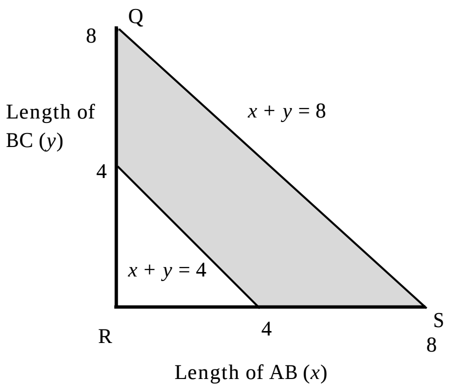


FIGURE 26.

The first inequality may be simplified as follows:

$x + y > 8 - x - y$
$2x + 2y > 8$
$x + y > 4$ (the shaded area in **Figure 26**).

The second inequality may be simplified as follows:

$x + (8 - x - y) > y$
$8 > 2y$
$4 > y$, or $y < 4$ (the shaded area shown in **Figure 27**).

The third inequality may be simplified as follows:

$y + (8 - x - y) > x$
$8 > 2x$
$4 > x$, or $x < 4$ (the shaded area shown in **Figure 28**).

The feasible region is the area of the common region defined by all three inequalities, as shown in **Figure 29**.

The probability that the three segments will make a triangle is found by comparing the feasible region to the sample space:

$$\frac{\text{Feasible region}}{\text{Sample space}} = \frac{\text{Area of Triangle EFG}}{\text{Area of Triangle QRS}}.$$

Area of triangle EFG $= \frac{1}{2}(4 \times 4) = 8$ sq in.

Area of triangle QRS $= \frac{1}{2}(8 \times 8) = 32$ sq in.

$$\frac{\text{Feasible region}}{\text{Sample space}} = \frac{8 \text{ sq in.}}{32 \text{ sq in.}} = 0.25.$$

The chance that two random points on segment AD will make three segments that form a triangle is 25%.

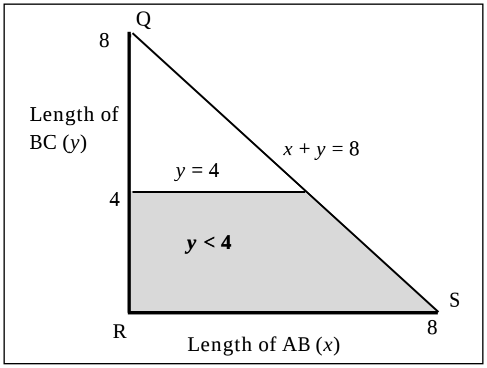


FIGURE 27.

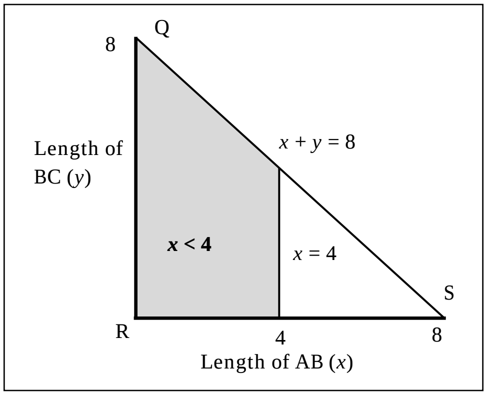


FIGURE 28.

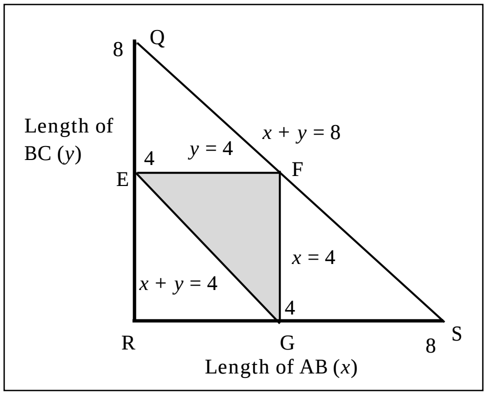


FIGURE 29.

EXERCISES

1. Ann and Candy want to improve the chance of meeting at the mall for lunch. They decide to wait 20 minutes before leaving. What is the probability they will meet between 12:00 noon and 1:00 PM if they change the waiting time to 20 minutes?

2. Robyn and River Banks own a joint checking account with a $200 balance. They go on a shopping trip and split up to buy each other an anniversary gift. If their checking account balance falls below $50, they are charged a penalty by the bank.
   a) What is the probability that their bank account will have more than $50 left in it after each of them writes a check for their purchase ?

   b) What is the probability that they will not be overdrawn after each of them has written a check?

3. A state police officer patrols Highway 8 from the state police post to Knoware, which is located 40 miles due east. Another state police officer patrols Highway 8 from the post to Someware, which is located 40 miles due west. Their radios can broadcast a maximum distance of only 20 miles.

a) Find the probability that the officers cannot communicate with each other. Assume that they cannot relay calls through the post.

b) Find the probability that the officers can communicate if they can relay calls through the post.

EXERCISES

4. If you break a piece of uncooked spaghetti into three pieces of random length, what is the probability that the three pieces will form a triangle?

5. In the first example in this section, what is the probability that Gail Force will run out of fuel on the way to her destination today?

6. Terry Dactyl is building a flower garden. At one end of the garden she plans to enclose a bed of petunias in a triangle of boards. Terry already has a board that is 4 feet long, which she does not intend to cut. When Terry goes down to the local hardware store, she sees a bin full of leftover boards, with lengths up to 7 feet. A clerk offers to help, and Terry tells the clerk to pick any two boards for her. What is the probability that the clerk selects two boards which will make a triangular flower bed with the 4-foot board Terry already has?

INVESTIGATION 5.

For this investigation you will need to gather data from people who are not in your class, such as family members, other students in your school, or your neighbors. Work in a group of 3 to 5 students. Each group member should collect data from five people. Ask each person to break a piece of spaghetti into three pieces. Once the spaghetti is broken, record whether the three pieces will form a triangle. Combine your data with the data of your group members to determine what percentage of all of the group could form a spaghetti triangle. If your results do not agree with the predicted results from Exercise 4 in Section 4, explain why you think the results are different.

SECTION 5

Bertrand's Paradox

In many problems the definition of a random occurrence will lead to surprising solutions. French mathematician Joseph L. F. Bertrand (1822–1900) found what is called Bertrand's Paradox, in what appears to be a very simple problem. (In mathematics, a paradox is a statement of apparently contradictory facts. The statement may nevertheless be true.) The paradox arises from different definitions of random occurrences. Bertrand proposed the following problem:

What is the probability that a random chord in a circle is longer than the side of an inscribed equilateral triangle?

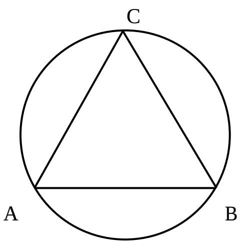


FIGURE 30.

Figure 30 shows an inscribed equilateral triangle in a circle of radius *r*.

The chords in the circle maybe thought of as line segments with ends randomly located on the circumference. The sample space for all chords is any chord on the entire circumference, or all 360° of the circumference.

The chords that are longer than side AC, the feasible region, also may be thought of as having endpoints randomly located anywhere along the the circumference of the circle.

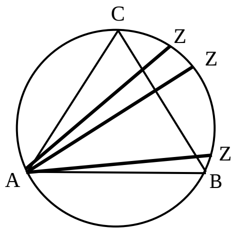


FIGURE 31.

For example, a chord such as AZ is longer than side AC if the location of Point Z is on the circle between Point C and Point B. Arc BC is the feasible region. Arc BC is an inscribed arc of A. Since triangle ABC is equilateral, m A = 60°, and the measure of arc BC = 120°. See **Figure 31**.

To find the probability that chord AZ is longer than side AC, we will compare the feasible region to the sample space.

$$\frac{\text{Feasible region}}{\text{Sample space}} = \frac{120°}{360°} = \frac{1}{3} = 0.3\overline{3}.$$

The chance that a random chord is longer than the side of an inscribed equilateral triangle is about 33%.

Notice that in this solution to Bertrand's problem we defined a random chord as one defined by endpoints anywhere along the circumference. What if the random chord is defined differently? Suppose a random chord is defined as having its midpoint located randomly anywhere inside the circle. **Figure 32** shows this possibility. Each of these chords is defined by its midpoint.

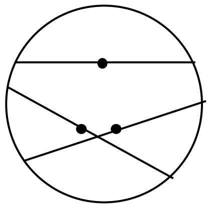


FIGURE 32.

What is the probability that a chord defined by a randomly chosen midpoint will be longer than the side of an inscribed equilateral triangle?

The sample space for all such chords is the measure of the entire circle, since the midpoint of any random chord could be anywhere in the circle. For a circle of radius r units, the sample space is πr^2 square units.

In a circle of radius r, the side of an inscribed equilateral triangle is $r\sqrt{3}$, as we shall prove. In **Figure 33**, OM is the perpendicular bisector of side AB, and mOB = r. Since triangle ABC is equilateral, m AOB = 120°. Thus m MOB = 60°. Triangle MOB is a 30°-60°-90° triangle, and mMB = $\frac{\sqrt{3}}{2}r$. AB = 2 x mMB = $r\sqrt{3}$.

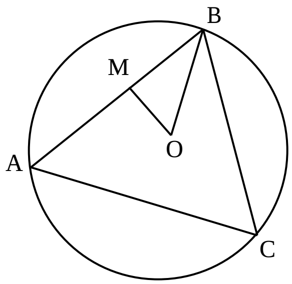


FIGURE 33.

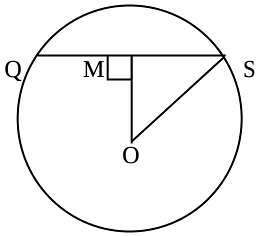
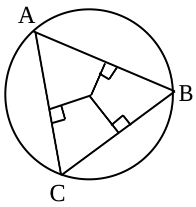


FIGURE 34.

To determine the feasible space, it is necessary to find all possible locations for the midpoints of chords longer than side AB. The triangle in **Figure 34** shows the location of a chord whose length is equal to the length of the side of the inscribed equilateral triangle.



AB = BC = BA

FIGURE 35.

Chord QS has a length of $r\sqrt{3}$. OS is a radius with length r. By the Pythagorean theorem, OM = $\frac{r}{2}$. Thus, any chord with a midpoint a distance of $\frac{r}{2}$ from the center of the circle will be equal in length to side AB (**Figure 35**).

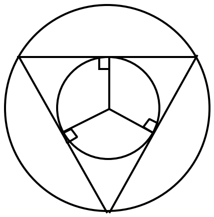


FIGURE 36.

If the midpoint of the chord is closer to the center of the circle, then the chord will be longer than $r\sqrt{3}$, which is longer than side AB. The feasible region for all midpoints is a circle with radius $\frac{r}{2}$, as shown in **Figure 36**.

The probability that such a chord is longer than the side of an equilateral triangle is found by comparing the feasible region to the sample space.

$$\frac{\text{Feasible region}}{\text{Sample space}} = \frac{\pi(\frac{r}{2})^2}{\pi r^2} = \frac{1}{4} = 0.25.$$

In the first case, the chance was about 33%. In this solution the chance is 25%. Both solutions are correct, and yet the answers are different. This is what is meant by a paradox. The reason for different answers lies with the two different ways of locating random chords. Which is correct? What do you think?

EXERCISES

1. What is the probability that a random chord in a circle is longer than the radius?

2. Can you see two ways to interpret “random chord” that lead to two different answers to Exercise 1?

Projects

1. French mathematician Blaise Pascal (1623–1662), a co-founder of Probability Theory with Pierre de Fermat, is known as “The Greatest Could-Have-Been” in mathematics history. Find out why and give a report to the class.

2. French mathematician Pierre de Fermat (1601–1665), a co-founder of Probability Theory with Blaise Pascal, is considered by historians to be the “Prince of Amateurs.” Find out why and give a report to the class.

3. French mathematician Pierre de Fermat (1601–1665), a co-founder of Probability Theory with Blaise Pascal, is the author of what is called Fermat’s Last Theorem. What is this theorem, and why is it making news in the twentieth century? Find out and give a report to the class.

4. French mathematician René Descartes (1596–1650), the inventor of coordinate geometry, left his native France to work for the Queen of Sweden. This was the worst mistake of his life. Find out why and give a report to the class.

5. Euclid (c. 300 BC) is considered the greatest geometer of all time. Some claim he never had an original thought in all his life. How could he be considered the “Father of Geometry” if none of his thoughts were original? Find out why and give a report to the class.

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