Throughout your life you are faced with decisions. As a student, you must decide which courses to take and how to divide your time among homework, activities, social events, and, perhaps, a job. As an adult, you will be faced with many new decisions, including how to vote in elections.

The decisions that people make can have important consequences. For example, Nielsen Media Research regularly polls individuals to learn what television programs they decide to watch. These viewer decisions determine whether a show will survive to another season. Because of the consequences of their work, organizations like Nielsen have a formidable responsibility: to combine the preferences of all the individuals in their survey into a single result and to do so in a way that is fair to all television programs.

• How are the wishes of many individuals combined to yield a single result?

• Do the methods for doing so always treat each choice fairly?

• If not, is it possible to improve on these methods?

This chapter examines a process that is fundamental to any democratic society: group decision making. As you progress through the chapter, you will consider several models that can be used by a group of people to conduct a vote and reach a decision. Election theory, the study of the voting process, is a recent example of the use of mathematical modeling in the social sciences.
Every democratic institution must have a process by which the preferences of individuals are combined to produce a group decision. For example, the preferences of individual voters must be combined in a fair way in order to select government officials.

An excellent way to begin an exploration of group decision making is to give the process a try. Therefore, in this lesson you will develop a model for combining preferences of the individuals in your class into a single result. Before you begin, a word of reassurance and a preview of things to come: Many important problems in election theory (and other topics in discrete mathematics) can be understood and solved without a lot of background knowledge, and mathematicians know that there is no single right solution for many of these problems.

Explore This

On a piece of paper write the names of the following soft drinks, in the order given:

- Coke
- Dr. Pepper
- Mountain Dew
- Pepsi
- Sprite
Lesson 1.1 • An Election Activity

Rank the soft drinks. That is, beside the name of the soft drink you like best, write “1.” Beside the name of your next favorite soft drink, write “2.” Continue until you rank all five.

As directed by your instructor, collect the ballots from all members of your class and share the results by, for example, writing all ballots on a chalkboard. Since everyone has written the soft drinks in the same order, you should be able to record quickly only the digits from each ballot.

Your task in this activity is to devise a method of combining the rankings of all the individuals in your class into a single ranking for the entire class. Your method should produce a first-, second-, third-, fourth-, and fifth-place soft drink.

If you are working in a small group, your group should agree on a single method. After everyone finishes, each group or individual should present the ranking to the class and describe the method used to obtain it. Clear communication of the method used to obtain a result is important in mathematics, so everyone should strive for clarity when making the presentation.

As each group (or individual) makes its presentation, record the ranking in your notebook for use in the following exercises.

Howard Bans
Distribution of Soft Drinks on County Property

Howard County Executive Ken Ulman moved Tuesday to ban the sale of high-sugar drinks such as soda in parks, libraries and other county properties and at county-sponsored events.

The sales and distribution ban — which mirrors efforts nationally and that may be adopted by Baltimore City — is aimed at reducing childhood obesity and raising awareness among parents and adults about the health hazards of sugary drinks.

Nationally, one of the most high-profile efforts aimed at tackling childhood obesity was the so-called super-size ban by the New York City Board of Health. That city plans to prohibit the sale of sugary drinks larger than 16 ounces by restaurants and other food vendors, starting in March.
Chapter 1 • Election Theory: Modeling the Voting Process

Exercises

1. Do all the group rankings produced in your class have the same soft drink ranked first? If not, which soft drink is ranked first most often?

2. Repeat Exercise 1 for the soft drink ranked second.

3. Repeat Exercise 1 for the soft drink ranked third.

4. Repeat Exercise 1 for the soft drink ranked fourth.

5. Repeat Exercise 1 for the soft drink ranked fifth.

6. Write a description of the method you used to achieve a group ranking. Make it clear enough that another person can use the method. You may want to break down the method into numbered steps.

7. Did anyone in your class use a method similar to yours? Explain why you think they are similar.

8. Did your method result in any ties? How can your method be modified to break ties?

9. Mathematicians often find it convenient to represent a situation in a compact way. A good representation conveys the essential information about a situation. In election theory, a preference schedule is sometimes used to represent the preferences of one or more individuals. The following preference schedule displays four choices, called A, B, C, and D. It indicates that the individual whose preference it represents ranks B first, C second, D third, and A fourth.

Since there are often several people who have the same preferences, mathematicians write the number of people or the percentage of people who expressed that preference under the schedule. The preferences in a group of 26 people are represented by the preference schedules at the top of the next page.
Lesson 1.1 • An Election Activity

a. Apply the method you used to determine your class’s soft drink ranking to this set of preferences. List the first-, second-, third-, and fourth-place rankings that your method produces. If your method cannot be applied to this set of preferences, then explain why it cannot and revise it so that it can be used here.

b. Do you think the ranking your method produces is fair? If you worked in a group, do all members of your group think the result is fair? In other words, do the first-, second-, third-, and fourth-place rankings seem reasonable, or are there reasons that one or more of the rankings seem unfair? Explain.

c. Would preference schedules be a useful way to represent the individual preferences for soft drinks among the members of your class? Explain.

10. When your class members voted on soft drinks, they ranked them from first through fifth. A ballot that allows voters to rank the choices is called a preferential ballot. In most elections in the United States, preferential ballots are not used. Do you think preferential ballots are a good idea? Explain.

11. There are three choices in a situation that uses preferential ballots. Call the choices A, B, and C. The figure at the top of the next page gives the six possible preferences that a voter can express.
A fourth choice, D, enters the picture. If D is attached to the bottom of each of the previous schedules, there are six schedules with D at the bottom. Similarly, there are six schedules with D third, six with D second, and six with D first, or a total of $4(6) = 24$ schedules. Thus, the total number of schedules with four choices is four times the total number of schedules with three choices.

a. There are 24 possible schedules with four choices. How many are there with five choices? With six choices?

b. Mathematicians use symbols to represent this relationship. The symbol $S_n$ represents the number of schedules when there are $n$ choices. You have seen that $S_n = nS_{n-1}$. Write an English translation of the mathematical sentence $S_n = nS_{n-1}$.

12. The mathematical sentence in Exercise 11b is a recurrence relation, a verbal or symbolic statement that describes how one number in a list is derived from the previous number (or numbers). Since recurrence relations are an important part of discrete mathematics, your experience with them begins in this lesson.

For example, suppose the first number in a list is 7 and a recurrence relation states that to obtain any number in the list, add 4 to the previous number. Then the second number is $7 + 4$, or 11. This recurrence relation is stated symbolically as $T_n = 4 + T_{n-1}$.

Another example of a recurrence relation is $T_n = n + T_{n-1}$. Complete the following table for the recurrence relation $T_n = n + T_{n-1}$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n$</th>
<th>$n$</th>
<th>$n$</th>
<th>$n$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>$T_n$</td>
<td>3</td>
<td>2 + 3 = 5</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

13. Complete the following table for the recurrence relation $A_n = 3 + 2A_{n-1}$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n$</th>
<th>$n$</th>
<th>$n$</th>
<th>$n$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>$A_n$</td>
<td>4</td>
<td>3 + 2(4) = 11</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

14. State the recurrence relation $A_n = 3A_{n-1} - 4$ in words.
Lesson 1.2

Group-Ranking Models

If the soft drink data for your class are typical, you know that the problem of establishing a group ranking is not without controversy. Even among professionals, there isn’t agreement on the best way to do so. This lesson examines several common models for determining a group ranking from a set of voter preferences. As you examine these models, consider whether any of them are similar to the ones devised by members of your class in Lesson 1.1.

Consider the preferences of Exercise 9 of the previous lesson, which are shown again in Figure 1.1.

![Figure 1.1 Preferences of 26 voters.](image)

Many voting situations, such as elections in which there is only one office to fill, require the selection of a single winner. Although most such elections in the United States do not use a preferential ballot, they could. For example, in the set of preferences shown in Figure 1.1, choice A is ranked first on eight schedules, more often than any other choice.
If A wins on this basis, A is called the **plurality winner**. A plurality winner is based on first-place rankings only. The winner is the choice that receives the most votes. Note, however, that A is first only on about 30.8% of the schedules. If A is first on over half the schedules, A is a **majority winner**.

### Borda Models

Did anyone in your class determine the soft drink ranking by assigning points to the first, second, third, and fourth choice of each individual’s preference and obtaining a point total? If so, these groups used a type of **Borda count**.

The most common way that Borda methods are applied to a ranking of $n$ choices is by assigning $n$ points to a first-place ranking, $n-1$ to a second-place ranking, $n-2$ to a third-place ranking, ... and 1 point to a last-place ranking. The group ranking is established by totaling each choice’s points.

In the example of Figure 1.1, A is ranked first by 8 people and fourth by the remaining 18, so A’s point total is $8(4) + 18(1) = 50$. Similar calculations give totals of 83, 69, and 58 for B, C, and D, respectively, as summarized below.

- **A**: $8(4) + 5(1) + 6(1) + 7(1) = 50$
- **B**: $8(3) + 5(4) + 6(3) + 7(3) = 83$
- **C**: $8(2) + 5(3) + 6(4) + 7(2) = 69$
- **D**: $8(1) + 5(2) + 6(2) + 7(4) = 58$

In this case, the plurality winner does not fare well under a Borda system.
Runoff Models

Some elections in the United States and other countries require a majority winner. If there is no majority winner, a runoff election between the top two candidates is held. Runoff elections are expensive because of the cost of holding another election and time-consuming for voters because they require a second trip to the polls. However, if voters use a preferential ballot, both disadvantages can be avoided.

To conduct a runoff, determine the number of firsts for each choice. In the example of Figure 1.1, A is first eight times, B is first five times, C is first six times, and D is first seven times.

Eliminate all choices except the two with more first-place votes than the others: Choices B and C are eliminated; A and D are retained.

Now consider each preference schedule on which the eliminated choices are ranked first. Choice B is first on the second schedule. Of the two remaining choices, A and D, D is ranked higher than A, so these 5 votes are transferred to D. Similarly, the 6 votes from the third schedule are transferred to D. The totals are now 8 for A and 7 + 5 + 6 = 18 for D, and so D is the runoff winner (see Figure 1.2).

The runoff method eliminates all choices except the two with the most firsts:

A: 8     B: 5     C: 6     D: 7
(Eliminate B & C.)

The five votes for B are transferred to D, and the six votes for C are transferred to D.

A: 8     D: 7 + 5 + 6 = 18.

Figure 1.2 A runoff.
Sequential Runoff Models

Some elections are conducted by a runoff variant that eliminates only one choice at a time. However, if there are several choices and voters must vote again when one is eliminated, sequential runoff methods compound the disadvantages of runoff methods. As with the runoff method, if voters use a preferential ballot, they need vote only once.

In the example of Figure 1.1, B is eliminated first because it is ranked first the fewest times. The 5 first-place votes for B are transferred to C. The point totals are now 8 for A, 5 + 6 = 11 for C, and 7 for D.

There are three choices remaining. Now D's total is the smallest, so D is eliminated next. The 7 votes are transferred to the remaining choice that is ranked highest by these 7. Thus, C is given an additional 7 votes and so defeats A by 18 to 8 (see Figure 1.3).

![Sequential Runoff Diagram]

**Figure 1.3** A sequential runoff.

When a preferential ballot is used and computers process the data, sequential runoff results are obtained almost instantaneously. Thus, the term **instant runoff** is often applied to sequential runoffs.
Lesson 1.2 • Group-Ranking Models

Exercises

For Exercises 1–4, apply the methods as they are described in this lesson.

1. Find a plurality winner for the soft drink voting in your class. Is it also a majority winner? Explain.

2. Find a Borda winner for the soft drink voting in your class.

3. Find a runoff winner for the soft drink voting in your class.

4. Find a sequential runoff winner for the soft drink voting in your class.

5. The International Olympic Committee uses sequential runoff voting to choose Olympic sites. Since the committee is relatively small, separate rounds of voting are used rather than a single round with preferential ballots. There were three cities competing to host the 2014 winter games. This table summarizes the voting.

<table>
<thead>
<tr>
<th>City</th>
<th>First Place</th>
<th>Second Place</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sochi</td>
<td>34</td>
<td>51</td>
</tr>
<tr>
<td>Pyeongchang</td>
<td>36</td>
<td>47</td>
</tr>
<tr>
<td>Salzburg</td>
<td>25</td>
<td></td>
</tr>
</tbody>
</table>

a. Write a short summary of the voting, including the order in which the cities were eliminated.

b. Five cities competed to host the 2016 summer games. Read the news article about the voting and construct a similar table.

Rio to Stage 2016 Olympic Games

BBC
Oct. 2, 2009

Brazil will become the first South American country to host the Olympics after the city of Rio de Janeiro was chosen to stage the 2016 Games.

Chicago's early exit was a surprise, after bookmakers made them favorites. Chicago received only 18 of the 94 votes available in the first round poll of IOC delegates. Madrid came out top with 28, followed by Rio with 26.

In the second round, however, Rio almost secured the absolute majority needed to win outright, with 46 of the 95 votes cast. Madrid came a distant second with 29, while Tokyo was eliminated after receiving 20.

The final ballot saw Rio win by a comprehensive margin of 66 votes to 32.
6. For the example of Figure 1.1, find the percentage of voters that rank each choice first and last.

a. Enter the results in a table like this:

<table>
<thead>
<tr>
<th>Choice</th>
<th>First</th>
<th>Last</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b. On the basis of these percentages only, which choice do you think is most objectionable to voters? Least objectionable? Explain your answers.

c. Which choice do you think most deserves to be ranked first for the group? Explain your reasoning.

d. Give at least one argument against your choice.

7. The 2010 race for governor of Minnesota had three strong candidates. The following are the results from the general election.

<table>
<thead>
<tr>
<th>Candidate</th>
<th>Votes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mark Dayton</td>
<td>919,232</td>
</tr>
<tr>
<td>Tom Emmer</td>
<td>910,462</td>
</tr>
<tr>
<td>Tom Horner</td>
<td>251,487</td>
</tr>
<tr>
<td>Others</td>
<td>25,840</td>
</tr>
</tbody>
</table>

a. What percentage of the vote did the winner receive? Is the winner a majority winner?

b. What is the smallest percentage that a plurality winner can receive in a race with three candidates? Explain.

8. Determine plurality, Borda, runoff, and sequential runoff winners for the following set of preferences. Apply the methods as they are described in this lesson.

```
A (7) B (12) C (20) D (16)
D (16) D (20) B (12) D (16)
C (12) A (7) D (16) A (7)
B (7) C (16) A (12) B (7)
```
Lesson 1.2 • Group-Ranking Models

9. Borda models produce a complete group ranking, but the other models examined in this lesson determine a winner only. However, each of them can be extended to produce a complete group ranking.

a. Describe how a plurality model could be extended to determine a second, third, and so forth. Apply your extension to the example in Figure 1.1 and list the second, third, and fourth that it produces.

b. Describe how runoff models could be extended to determine a second, third, and so forth. Apply your extension to the example in Figure 1.1 and list the second, third, and fourth that it produces.

c. Describe how a sequential runoff model could be extended to determine a second, third, and so forth. Apply your extension to the example in Figure 1.1 and list the second, third, and fourth that it produces.

10. Each year the Heisman Trophy recognizes one of the country’s outstanding college football players. In 2012, Texas A & M freshman quarterback Johnny Manziel received the award. The results of the voting follow. Each voter selects a player to rank first, another to rank second, and another to rank third.

<table>
<thead>
<tr>
<th>Rank</th>
<th>Player</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Johnny Manziel</td>
<td>474</td>
<td>252</td>
<td>103</td>
<td>2,029</td>
</tr>
<tr>
<td>2</td>
<td>Manti Te’o</td>
<td>321</td>
<td>309</td>
<td>125</td>
<td>1,706</td>
</tr>
<tr>
<td>3</td>
<td>Collin Klein</td>
<td>60</td>
<td>197</td>
<td>320</td>
<td>894</td>
</tr>
<tr>
<td>4</td>
<td>Marqise Lee</td>
<td>19</td>
<td>33</td>
<td>84</td>
<td>207</td>
</tr>
<tr>
<td>5</td>
<td>Braxton Miller</td>
<td>3</td>
<td>29</td>
<td>77</td>
<td>144</td>
</tr>
<tr>
<td>6</td>
<td>Jadeveon Clowney</td>
<td>4</td>
<td>13</td>
<td>23</td>
<td>61</td>
</tr>
<tr>
<td>7</td>
<td>Jordan Lynch</td>
<td>3</td>
<td>8</td>
<td>27</td>
<td>52</td>
</tr>
<tr>
<td>8</td>
<td>Tavon Austin</td>
<td>6</td>
<td>4</td>
<td>21</td>
<td>47</td>
</tr>
<tr>
<td>9</td>
<td>Kenjon Barner</td>
<td>1</td>
<td>12</td>
<td>15</td>
<td>42</td>
</tr>
<tr>
<td>10</td>
<td>Jarvis Jones</td>
<td>1</td>
<td>10</td>
<td>18</td>
<td>41</td>
</tr>
</tbody>
</table>

a. How many points are awarded for a first-place vote? For a second-place vote? For a third-place vote?

b. Does the ranking produced by this model differ if a plurality model is used? Explain.
11. In most American runoff elections, voters do not use a preferential ballot and therefore must return to the polls to vote in the runoff. Voters in some countries, such as Ireland, use a preferential ballot and therefore go to the polls only once. Examine the vote totals in the two runoffs shown below. What do the totals tell you about the merits of preferential ballots in runoff elections?

**President of Ireland: 2011 Results**

<table>
<thead>
<tr>
<th></th>
<th>General Election</th>
<th>Runoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>Michael Higgins</td>
<td>701,101</td>
<td>1,007,104</td>
</tr>
<tr>
<td>Seán Gallagher</td>
<td>504,964</td>
<td>628,114</td>
</tr>
<tr>
<td>Martin McGuinness</td>
<td>243,030</td>
<td></td>
</tr>
<tr>
<td>Gay Mitchell</td>
<td>113,321</td>
<td></td>
</tr>
<tr>
<td>David Norris</td>
<td>109,469</td>
<td></td>
</tr>
<tr>
<td>Dana Scallon</td>
<td>51,220</td>
<td></td>
</tr>
<tr>
<td>Mary Davis</td>
<td>48,657</td>
<td></td>
</tr>
</tbody>
</table>

**U.S. House Louisiana District 3: 2012 Results**

<table>
<thead>
<tr>
<th></th>
<th>General Election</th>
<th>Runoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>Charles Boustany</td>
<td>139,123</td>
<td>58,820</td>
</tr>
<tr>
<td>Jeff Landry</td>
<td>93,527</td>
<td>37,764</td>
</tr>
<tr>
<td>Ron Richard</td>
<td>67,070</td>
<td></td>
</tr>
<tr>
<td>Bryan Barrilleaux</td>
<td>7,908</td>
<td></td>
</tr>
<tr>
<td>Jim Stark</td>
<td>3,765</td>
<td></td>
</tr>
</tbody>
</table>

**Michael Higgins Becomes Irish President**

_The Guardian_  
Oct. 29, 2011

The Irish Labour party's Michael D Higgins has been confirmed as the ninth president of the republic after winning a landslide victory in the most fractious campaign in the country's history.

The poet and campaigner gained a 56.8% share of the vote, putting him comfortably ahead of his rivals – Dragons' Den star Seán Gallagher, who came second, and former IRA commander Martin McGuinness, who ended in third place.

The result capped a two-day count of ballots to determine who would succeed Mary McAleese as Ireland's head of state.
12. In sequential runoffs, the number of choices on a given round is 1 less than the number of choices on the previous round. Let $C_n$ represent the number of choices after $n$ rounds and write this as a recurrence relation.

13. A procedure for solving a problem is called an algorithm. This lesson discusses various algorithms for obtaining a group decision from individual preferences. Algorithms are often written in numbered steps in order to make them easy to apply. The following is an algorithmic description of a runoff model as discussed in this lesson.

1. For each choice, determine the number of preference schedules on which the choice is ranked first.

2. Eliminate all choices except the two that are ranked first most often.

3. For each preference schedule, transfer the vote total to the remaining choice that ranks highest on that schedule.

4. Determine the vote total for the preference schedules on which each of the remaining choices is ranked first.

5. The winner is the choice ranked first on the most schedules.

a. Write an algorithmic description of the sequential runoff model discussed in this lesson.

b. Write an algorithmic description of the Borda model discussed in this lesson.

14. The number of first-, second-, third-, and fourth-place votes for each choice in an election can be described in a table, or matrix, as shown below.
The number of points that a choice receives for first, second, third, and fourth place can be written in a matrix, as shown below.

<table>
<thead>
<tr>
<th></th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
</tr>
</thead>
<tbody>
<tr>
<td>Points</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

A new matrix that gives Borda point totals for each choice can be computed by writing this matrix alongside the first, as shown below.

\[
\begin{bmatrix}
4 & 3 & 2 & 1 \\
20 & 10 & 27 & 0 \\
0 & 12 & 0 & 45 \\
25 & 0 & 20 & 12 \\
12 & 35 & 10 & 0 \\
\end{bmatrix}
\]

Entries for a new matrix can be computed in this way: Multiply each entry of the first matrix by the corresponding entry in the first column of the second matrix and find the sum of these products.

\[4(20) + 3(0) + 2(25) + 1(12) = 142\]

This number is the first entry in a new matrix that gives Borda point totals for choices A, B, C, and D:

\[
\begin{bmatrix}
A & B & C & D \\
Point totals: [142 & & & ]
\end{bmatrix}
\]

a. Calculate the remaining entries of the new matrix.

b. If you have this new matrix but not the voter preference schedules, by which methods is it possible to determine the winner? Explain.

**Computer/Calculator Explorations**

15. Enter the soft drink preferences of your class members into the election machine computer program that accompanies this book. Compare the results given by the computer to your answers to the first four exercises of this lesson. Resolve any discrepancies.
Projects

16. Write a short report on the history of any of the models discussed in this lesson. Look into the lives of people who were influential in developing the model. Discuss factors that led them to propose the model.

17. Find at least two examples of group-ranking models that are currently used somewhere in the world but not discussed in this lesson. Describe how the group ranking is determined. Compare each new model with those described in this lesson. What are some advantages and disadvantages of each new model?

18. Select one or more countries that are not discussed in this lesson and report on the models they use to conduct elections.

Bayern Munich Stays Best Team in AP Global Poll

Associated Press
April 30, 2013

After dominating the first legs of their Champions League semifinals, it's no surprise to see Bayern Munich and Borussia Dortmund leading the way in the latest Associated Press global football poll.

Bayern pipped Dortmund to be voted the world's top team for the fifth straight week, with its 4-0 hammering of Barcelona impressing the AP’s panel of 15 journalists.

AP Global Football Rankings for the week ending April 29.

Based on 15 voters, using 10 points for first, nine for second, one for bottom place. Previous rankings in parentheses.

1. Bayern Munich (1), 147
2. Borussia Dortmund (7), 136
3. Juventus (2), 71
5. Chelsea, 58
6. Liverpool, 50
7. Real Madrid (4), 36
8. Paris Saint-Germain (6), 30
9. Real Sociedad
10. Barcelona (5), 24
Different models for finding a group ranking can give different results. This fact led the Marquis de Condorcet to propose that a choice that could obtain a majority over every other choice should be ranked first for the group.

Again consider the set of preference schedules used in the previous lesson (see Figure 1.4).

To examine these data for a Condorcet winner, compare each choice with every other choice. For example, begin by comparing A with B, then with C, and finally with D. Notice in Figure 1.4 that A is ranked higher than B on 8 schedules and lower on 18. (An easy way to see this is to cover C and D on all the schedules.) Because A cannot obtain a majority against B, A cannot be a Condorcet winner. Therefore, there is no need to check to see if A can beat C or D.
Now consider B. You have already seen that B beats A, so begin by comparing B with C. B is ranked higher than C on $8 + 5 + 7 = 20$ schedules and lower than C on 6.

Now compare B with D. B is ranked higher than D on $8 + 5 + 6 = 19$ schedules and lower than D on 7. Therefore, B has a majority over each of the other choices and so is a Condorcet winner.

Since B is a Condorcet winner, it is unnecessary to make comparisons between C and D. Although all comparisons do not always have to be made, it can be helpful to organize them in a table:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>L</td>
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<tr>
<td>B</td>
<td>W</td>
<td>W</td>
<td>W</td>
<td>L</td>
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<tr>
<td>C</td>
<td>W</td>
<td>L</td>
<td>W</td>
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<tr>
<td>D</td>
<td>W</td>
<td>L</td>
<td>L</td>
<td>L</td>
</tr>
</tbody>
</table>

To see how a choice does in one-on-one contests, read across the row associated with that choice. You see that A, for example, loses in one-on-one contests with B, C, and D.

Although Condorcet’s model may sound ideal, it sometimes fails to produce a winner. Consider the set of schedules shown in Figure 1.5.

\[\begin{array}{c|c|c|c|c}
A & B & C & D \\
\hline
A & L & L & L \\
B & W & W & W \\
C & W & L & W \\
D & W & L & L \\
\end{array}\]

**Figure 1.5** Preferences of 60 voters.

Notice that A is preferred to B on 40 of the 60 schedules but that A is preferred to C on only 20. Although C is preferred to A on 40 of the 60, C is preferred to B on only 20. Therefore there is no Condorcet winner.
You might expect that if A is preferred to B by a majority of voters and B is preferred to C by a majority of voters, then a majority of voters prefer A to C. But the example shows that this need not be the case.

In other mathematics classes you have learned that many relationships are transitive. The relation “greater than” (>), for example, is transitive because if \( a > b \) and \( b > c \), then \( a > c \).

You have just seen that group-ranking models may violate the transitive property. Because this intransitivity seems contrary to intuition, it is known as a paradox. This particular paradox is sometimes referred to as the Condorcet paradox. There are other paradoxes that can occur with group-ranking models, as you will see in this lesson’s exercises.

**Exercises**

1. Find a Condorcet winner in the soft drink ballot your class conducted in Lesson 1.1.

2. Propose a method for resolving situations in which there is no Condorcet winner.

3. In a system called pairwise voting, two choices are selected and a vote taken. The loser is eliminated, and a new vote is taken in which the winner is paired against a new choice. This process continues until all choices but one have been eliminated. An example of the use of pairwise voting occurs in legislative bodies in which bills are considered two at a time. The choices in the set of preferences shown in the following figure represent three bills being considered by a legislative body.

   ![Figure](image)

   a. Suppose you are responsible for deciding which two bills appear on the agenda first. If you strongly prefer bill C, which two bills would you place on the agenda first? Why?

   b. Is it possible to order the voting so that some other choice wins? Explain.
4. A panel of sportswriters is selecting the best football team in a league, and the preferences are distributed as follows.

   A   B   C
   B   A   B
   C   C   A
   52  38  10

   a. Determine a best team using a 3-2-1 Borda count.

   b. The 38 who rank B first and A second decide to lie in order to improve the chances of their favorite and so rank C second. Determine the winner using a 3-2-1 Borda count.

5. When people decide to vote differently from the way they feel about the choices, they are said to be voting insincerely. People are often encouraged to vote insincerely because they have some idea of an election’s result beforehand. Explain why such advance knowledge is possible.

6. Many political elections in the United States are decided with a plurality model. Construct a set of preferences with three choices in which a plurality model could encourage insincere voting. Identify the group of voters that might be encouraged to vote insincerely and explain the effect of their insincere voting on the election outcome.

7. Many people consider plurality models flawed because they can produce a winner that a majority of voters do not like.

   a. What percentage of voters ranks the plurality winner last in the preferences shown below?

   A   C   B   D
   B   B   D   E
   D   E   E   C
   C   D   C   B
   E   A   A   A
   20  18  15  12
b. Runoffs are sometimes used to avoid the selection of a controversial winner. Is a runoff winner an improvement over a plurality winner in this set of preferences? Explain.

c. Do you consider a sequential runoff winner an improvement over plurality and runoff winners? Explain.

8. a. Use a runoff to determine a winner in the following set of preferences.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>C</th>
<th>B</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>B</td>
<td>C</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>B</td>
<td>A</td>
<td>C</td>
</tr>
<tr>
<td></td>
<td>38</td>
<td>30</td>
<td>25</td>
<td>7</td>
</tr>
</tbody>
</table>

b. In some situations, votes are made public. For example, people have the right to know how their elected officials vote on issues. Suppose these schedules represent such a situation. Because they expect to receive some favors from the winner and because they expect A to win, the seven voters associated with the last schedule decide to change their preferences from

<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>to</th>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td></td>
<td>C</td>
</tr>
</tbody>
</table>

and to “go with the winner.” Conduct a new runoff and determine the winner.

c. Explain why the results are a paradox.
Lesson 1.3 • More Group-Ranking Models and Paradoxes

9. a. Use a 4-3-2-1 Borda count to find a group ranking for the following set of preferences.

   ![Borda count diagram]

   b. These preferences represent the ratings of four college athletic teams, and team C has been disqualified because of a recruiting violation. Write the schedules with team C removed and use a 3-2-1 Borda count to determine a group ranking.

c. Explain why these results are a paradox.

10. For each voting model discussed in this and the previous lesson (plurality, Borda, runoff, sequential runoff, and Condorcet), write a brief summary. Include at least one example of why the model can lead to unfair results.

11. In theory, Condorcet models require that each choice be compared with every other one, although in practice many of the comparisons do not have to be made in order to determine the winner. Consider the number of comparisons when every possible comparison is made.

   Mathematicians sometimes find it helpful to represent the choices and comparisons visually. If there are only two choices, a single comparison is all that is necessary. In the diagram that follows, a point, or vertex, represents a choice, and a line segment, or edge, represents a comparison.

   ![Diagram with edge between A and B]

   a. Add a third choice, C, to the diagram. Connect it to A and to B to represent the additional comparisons. How many new comparisons are there? What is the total number of comparisons?
b. Add a fourth choice, D, to the diagram. Connect it to each of A, B, and C. How many new comparisons are there, and what is the total number of comparisons?

c. Add a fifth choice to the diagram and repeat. Then add a sixth choice and repeat. Complete the following table.

<table>
<thead>
<tr>
<th>Number of Choices</th>
<th>Number of New Comparisons</th>
<th>Total Number of Comparisons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

12. Let \( C_n \) represent the total number of comparisons necessary when there are \( n \) choices. Write a recurrence relation that expresses the relationship between \( C_n \) and \( C_{n-1} \).

13. U.S. College Hockey Online (USCHO) has several ranking systems. In a system called pairwise ranking, USCHO compares each team to every other team. In each comparison, the team that compares favorably to the other is awarded a point. The team with the most points is ranked first. Consider a simple version of this system in a league with 6 teams, A, B, C, D, E, and F. The following table shows the results of the comparisons. An X in a team’s row indicates that it won the comparison with the team at the top of the column.

<table>
<thead>
<tr>
<th>Rank</th>
<th>Team</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Quinnipiac</td>
<td>30</td>
</tr>
<tr>
<td>2</td>
<td>Minnesota</td>
<td>29</td>
</tr>
<tr>
<td>3</td>
<td>Massachusetts-Lowell</td>
<td>27</td>
</tr>
<tr>
<td>4</td>
<td>Notre Dame</td>
<td>26</td>
</tr>
<tr>
<td>5t</td>
<td>Miami</td>
<td>25</td>
</tr>
<tr>
<td>5t</td>
<td>Boston College</td>
<td>25</td>
</tr>
<tr>
<td>7</td>
<td>New Hampshire</td>
<td>24</td>
</tr>
<tr>
<td>8</td>
<td>North Dakota</td>
<td>23</td>
</tr>
<tr>
<td>9t</td>
<td>Denver</td>
<td>20</td>
</tr>
<tr>
<td>9t</td>
<td>Niagara</td>
<td>20</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
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<tr>
<td>B</td>
<td></td>
<td>X</td>
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<td>D</td>
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<td>X</td>
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<td>F</td>
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</tbody>
</table>

Examination Copy © COMAP Inc. Not For Resale
a. Find two pairwise comparisons in this table that demonstrate the transitive property. Find two comparisons that demonstrate a violation of the transitive property.

b. If each team receives a point for each comparison that it wins, find a group ranking for these teams.

c. Suggest a modification to the point system that is advantageous to team F.

d. Suppose team D drops out of the league. What effect does this have on the rankings you found in part b?

**Computer/Calculator Explorations**

14. Use the preference schedule program that accompanies this book to find a set of preferences with at least four choices that demonstrates the same paradox found in Exercise 8, but when sequential runoff is used.

15. Use the preference schedule program to enter several schedules with five choices. Use the program’s features to alter your data in order to produce a set of preferences with several different winners. Can you find a set of preferences with five choices and five different winners? If so, what is the minimum number of schedules with which this can be done? Explain.

**Projects**

16. Research and report on paradoxes in mathematics. Try to determine whether the paradoxes have been satisfactorily resolved.

17. Research and report on paradoxes outside mathematics. In what way have these paradoxes been resolved?

18. Select an issue of current interest in your community or school that involves more than two choices. Have each member of your class vote by writing a preference schedule. Compile the preferences and determine winners by five different methods.

19. Investigate the contributions of Charles Dodgson (Lewis Carroll) to election theory. Was he responsible for any of the group-ranking procedures you have studied? What did he suggest doing when Condorcet models fail to produce a winner?
20. Investigate the system your school uses to determine academic rankings of students. Is it similar to any of the group-ranking procedures you studied? If so, could it suffer from any of the same problems? Propose another system and discuss why it might be better or worse than the one currently in use.

21. Investigate elections in your school (class officers, officers of organizations, homecoming royalty, and so forth). Report on the type of voting and the way winners are chosen. Recommend alternative methods and explain why you think the methods you recommend are fairer.

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**Switzerland Jumps to #1 as Brand US Falls Further into Decline**

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Top 25 country brands of 2012–13:

1. Switzerland (+1 from 2012)
2. Canada (–1)
3. Japan (+1)
4. Sweden (+3)
5. New Zealand (–2)
6. Australia (–1)
7. Germany (+4)
8. United States (–2)
9. Finland (–1)
10. Norway (+2)
11. United Kingdom (+2)
12. Denmark (+3)
13. France (–4)
14. Singapore (+2)
15. Italy (–5)
16. Maldives (+2)
17. Austria (0)
18. Netherlands (+5)
19. Spain (–5)
20. Mauritius (+2)
21. Ireland (–1)
22. Iceland (–3)
23. United Arab Emirates (+2)
24. Bermuda (–3)
25. Costa Rica (–1)
Paradoxes, unfair results, and insincere voting are some of the problems that have caused people to look for better models for reaching group decisions. In this lesson you will learn of some recent and important work that has been done in attempts to improve the group-ranking process.

First, consider an example involving pairwise voting.

Ten representatives of the language clubs at Central High School are meeting to select a location for the clubs’ annual joint dinner. The committee must choose among a Chinese, French, Italian, or Mexican restaurant (see Figure 1.6).

Racquel says that because the last two dinners were at Mexican and Chinese restaurants, this year’s dinner should be at either an Italian or a French restaurant. The group votes 7 to 3 in favor of the Italian restaurant.
Martin, who doesn’t like Italian food, says that the community’s newest Mexican restaurant has an outstanding reputation. He proposes that the group choose between Italian and Mexican. The other members agree and vote 7 to 3 to hold the dinner at the Mexican restaurant.

Sarah, whose parents own a Chinese restaurant, says that she can obtain a substantial discount for the event. The group votes between the Mexican and Chinese restaurants and selects the Chinese by a 6 to 4 margin.

Look carefully at the group members’ preferences. Note that French food is preferred to Chinese by all, yet the voting selected the Chinese restaurant!

In 1951, paradoxes such as this led Kenneth Arrow, a U.S. economist, to formulate a list of five conditions that he considered necessary for a fair group-ranking model. These fairness conditions today are known as Arrow’s conditions.

One of Arrow’s conditions says that if every member of a group prefers one choice to another, then the group ranking should do the same. According to this condition, the choice of the Chinese restaurant when all members rated French food more favorably than Chinese is unfair. Thus, Arrow considers pairwise voting a flawed group-ranking method.

Arrow inspected common models for determining a group ranking for adherence to his five conditions. He also looked for new models that would meet all five. After doing so, he arrived at a surprising conclusion.

In this lesson’s exercises, you will examine a number of group-ranking models for their adherence to Arrow’s conditions. You will also learn Arrow’s surprising result.

Mathematician of Note

Kenneth Arrow (1921– )

Kenneth Arrow received a degree in mathematics before turning to economics. His use of mathematical methods in election theory brought him worldwide recognition.
Lesson 1.4  •  Arrow’s Conditions and Approval Voting

Arrow’s Conditions

1. Nondictatorship: The preferences of a single individual should not become the group ranking without considering the preferences of the others.
2. Individual Sovereignty: Each individual should be allowed to order the choices in any way and to indicate ties.
3. Unanimity: If every individual prefers one choice to another, then the group ranking should do the same. (In other words, if every voter ranks A higher than B, then the final ranking should place A higher than B.)
4. Freedom from Irrelevant Alternatives: If a choice is removed, the order in which the others are ranked should not change. (The choice that is removed is known as an irrelevant alternative.)
5. Uniqueness of the Group Ranking: The method of producing the group ranking should give the same result whenever it is applied to a given set of preferences. The group ranking should also be transitive.

Exercises

1. Your teacher decides to order soft drinks for your class on the basis of the soft drink vote conducted in Lesson 1.1 but, in so doing, selects the preference schedule of a single student (the teacher’s pet). Which of Arrow’s conditions are violated by this method of determining a group ranking?

2. Instead of selecting the preference schedule of a favorite student, your teacher places all the individual preferences in a hat and draws one. If this method were repeated, would the same group ranking result? Which of Arrow’s conditions does this method violate?
3. Do any of Arrow’s conditions require that the voting process include a secret ballot? Is a secret ballot desirable in all group-ranking situations? Explain.

4. Examine the paradox demonstrated in Exercise 9 of Lesson 1.3 on page 23. Which of Arrow’s conditions are violated?

5. Construct a set of preference schedules with three choices, A, B, and C, showing that the plurality method violates Arrow’s fourth condition. In other words, construct a set of preferences in which the outcome between A and B depends on whether C is on the ballot.

6. You have seen situations in which insincere voting occurs. Do any of Arrow’s conditions state that insincere voting should not be part of a fair group-ranking model? Explain.

7. Suppose that there are only two choices in a list of preferences and that the plurality method is used to decide the group ranking. Which of Arrow’s conditions could be violated? Explain.

8. A group of voters have the preferences shown in the following figure.

```
  A  A  B  C  D
  C  D  C  D  B
  D  B  A  B  C
  B  C  D  A  A

a. Use plurality, Borda, runoff, sequential runoff, and Condorcet models to find winners.

b. Investigate this set of preferences for violation of Arrow’s fourth condition. That is, can a choice change a winner by withdrawing?
9. Read the news article about the Google search engine.

a. Does the transitive property apply to individual Google voting? That is, if site A casts a Google vote for site B and site B casts a Google vote for site C, then must site A cast a Google vote for site C?

b. Does the transitive property apply to the Google ranking system? That is, if site A ranks higher than site B and site B ranks higher than site C, then must site A rank higher than site C? Explain.

10. After failing to find a group-ranking model for three or more choices that always obeyed all his fairness conditions, Arrow began to suspect that such a model does not exist. He applied logical reasoning and proved that no model, known or unknown, can always obey all five conditions. In other words, any group-ranking model violates at least one of Arrow’s conditions in some situations.

Arrow’s proof demonstrates how mathematical reasoning can be applied to areas outside mathematics. This and other achievements earned Arrow the 1972 Nobel Prize in economics.

Although Arrow’s work means that a perfect group-ranking model will never be devised, it does not mean that current models cannot be improved. Recent studies have led some experts to recommend approval voting.

Is Google Page Rank Still Important?

Search Engine Journal
October 6, 2004

Since 1998 when Sergey Brin and Larry Page developed the Google search engine, it has relied on the Page Rank Algorithm. Google’s reasoning behind this is, the higher the number of inbound links ‘pointing’ to a Website, the more valuable that site is, in which case it would deserve a higher ranking in its search results pages.

If site ‘A’ links to site ‘B’, Google calculates this as a ‘vote’ for site B. The higher the number of votes, the higher the overall value for site ‘B’. In a perfect world, this would be true. However, over the years, some site owners and webmasters have abused the system, implementing some ‘link farms’ and linking to Websites that have little or nothing to do with the overall theme or topic presented in their sites.
In approval voting, you may vote for as many choices as you like, but you do not rank them. You mark all those of which you approve. For example, if there are five choices, you may vote for as few as none or as many as five.

a. Write a soft drink ballot like the one you used in Lesson 1.1. Place an “X” beside each of the soft drinks you find acceptable. At the direction of your instructor, collect ballots from the other members of your class. Count the number of votes for each soft drink and determine a winner.

b. Determine a complete group ranking.

c. Is the approval winner the same as the plurality winner in your class?

d. How does the group ranking in part b compare with the Borda ranking that you found in Lesson 1.1?

11. Examine Exercise 4 of Lesson 1.3 on page 21. Would any members of the panel of sportswriters be encouraged to vote insincerely if approval voting were used? Explain.

12. What is the effect on a group ranking of casting approval votes for all choices? Of casting approval votes for none of the choices?

13. The voters whose preferences are represented below all feel strongly about their first choices but are not sure about their second and third choices. They all dislike their fourth and fifth choices. Since the voters are unsure about their second and third choices, they flip coins to decide whether to give approval votes to their second and third choices.

\[
\begin{align*}
&\text{A} &\text{D} &\text{E} \\
&\text{B} &\text{B} &\text{C} \\
&\text{C} &\text{E} &\text{B} \\
&\text{D} &\text{C} &\text{D} \\
&\text{E} &\text{A} &\text{A} \\
\end{align*}
\]

22 20 18

a. Assuming the voters’ coins come up heads half the time, how many approval votes would you expect each of the five choices to get? Explain your reasoning.

b. Do the results seem unfair to you in any way? Explain.
14. Approval voting offers a voter many choices. If there are three candidates for a single office, for example, the plurality system offers the voter four choices: vote for any one of the three candidates or for none of them. Approval voting permits the voter to vote for none, any one, any two, or all three.

To investigate the number of ways in which you can vote under approval voting, consider a situation with two choices, A and B. You can represent voting for none by writing \{\}, voting for A by writing \{A\}, voting for B by writing \{B\}, and voting for both by writing \{A, B\}.

(a) List all the ways of voting under an approval system when there are three choices.

(b) List all the ways of voting under an approval system when there are four choices.

(c) Generalize the pattern by letting $V_n$ represent the number of ways of voting under an approval system when there are $n$ choices and writing a recurrence relation that describes the relationship between $V_n$ and $V_{n-1}$.

15. Listing all the ways of voting under the approval system can be difficult if not approached systematically. The following algorithm describes one way to find all the ways of voting for two choices. The results are shown applied to a ballot with five choices, A, B, C, D, and E.

1. List all choices in order in List 1.

2. Draw a line through the first choice in List 1 that doesn’t already have a line drawn through it. Write this choice as many times in List 2 as there are choices in List 1 without lines through them.

3. Beside each item you wrote in List 2 in step 2, write a choice in List 1 that does not have a line through it.

4. Repeat steps 2 and 3 until each choice has a line through it. The items in the second list show all the ways of voting for two items.

Write an algorithm that describes how to find all the ways of voting for three choices. You may use the results of the previous algorithm to begin the new one.
16. Many patterns can be found in the various ways of voting when the approval system is used. The following table shows the number of ways of voting for exactly one item when there are several choices on the ballot. For example, in Exercise 14, you listed all the ways of voting when there are three choices on the ballot. Three of these, \{A\}, \{B\}, and \{C\}, are selections of one item.

<table>
<thead>
<tr>
<th>Number of Choices on the Ballot</th>
<th>Number of Ways of Selecting Exactly One Item</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

Complete the table.

17. Let \(V_1^n\) represent the number of ways of selecting exactly one item when there are \(n\) choices on the ballot and write a recurrence relation that expresses the relationship between \(V_1^n\) and \(V_1^{n-1}\).

18. The following table shows the number of ways of voting for exactly two items when there are from one to five choices on the ballot. For example, your list in Exercise 14 shows that when there are three choices on the ballot, there are three ways of selecting exactly two items: \{A, B\}, \{A, C\}, and \{B, C\}.

<table>
<thead>
<tr>
<th>Number of Choices on the Ballot</th>
<th>Number of Ways of Selecting Exactly Two Items</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

Complete the table.

19. Let \(V_2^n\) represent the number of ways of selecting exactly two items when there are \(n\) choices on the ballot and write a recurrence relation that expresses the relationship between \(V_2^n\) and \(V_2^{n-1}\). Can you find more than one way to do this?
**Computer/Calculator Explorations**

20. Design a computer program that lists all possible ways of voting when approval voting is used. Use the letters A, B, C, . . . to represent the choices. The program should ask for the number of choices and then display all possible ways of voting for one choice, two choices, and so forth.

**Projects**

21. Investigate the number of ways of voting under the approval system for other recurrence relations (see Exercises 16 through 19). For example, in how many ways can you vote for three choices, four choices, and so forth?

22. Arrow’s result is an example of an impossibility theorem. Investigate and report on other impossibility theorems.

23. Research and report on Arrow’s theorem. The theorem is usually proved by an indirect method. What is an indirect method? How is it applied in Arrow’s case?

24. In approval voting voters apply an approve or disapprove rating to each choice. Thus, approval voting is a rating system—not a ranking system. In 2007, Michel Balinski and Rida Laraki proposed another type of rating system called majority judgment, in which voters are allowed more than two ratings. Research and report on majority judgment. What are its advantages and disadvantages over other voting models?
Lesson 1.5

Weighted Voting and Voting Power

The first four lessons of this chapter examined situations in which all voters are considered equals. This lesson examines situations in which some voters have more votes than other voters.

Public Hearing Set for Madison County Weighted Voting

Oneida Daily Dispatch
April 11, 2013

The Madison County Board of Supervisors will hear public comments over upgrading weighted voting totals.

In Madison County, supervisors vote based on a weighted formula calculated by the population of their towns. That voting configuration hasn’t been updated since 2002. Using population data from the 2010 Census, the number has been recalculated.

Under the proposed law, there are to be 1500 votes total. The votes are not divisible and are to be cast as one lump sum by each town. The county uses the Banzhaf power index to determine how many votes each supervisor has.
Members of legislative bodies such as the United States Congress and county boards represent districts. The constitutional principle of one person, one vote requires that such districts be approximately the same size. Thus, every 10 years, district lines are redrawn to reflect changes in population. But redrawing boundaries can be difficult for many reasons. Some legislative bodies try to resolve difficulties by adopting a system in which some votes carry more weight than others.

Consider a simple example. A small high school has 110 students. Because of recent growth in the size of the community, the sophomore class is quite large. It has 50 members, and the junior and senior classes each have 30 members.

The school’s student council is composed of a single representative from each class. Each of the three members is given a number of votes proportionate to the size of the class represented. Accordingly, the sophomore representative has five votes, and the junior and senior representatives each have three. The passage of any issue that is before the council requires a simple majority of six votes.

The student council’s voting model is an example of weighted voting. Weighted voting occurs whenever some members of a voting body have more votes than others have.

In recent years, several people have questioned whether weighted voting is fair. Among them is John Banzhaf III, a law professor at George Washington University who has initiated several legal actions against weighted voting procedures used in local government.

To understand Banzhaf’s objection to weighted voting, consider the number of ways that voting on an issue could occur in the student council example.

It is possible that an issue is favored by none of the members, one of them, two of them, or all three. In which cases does an issue pass? The following list gives all possible ways of voting for an issue and the associated number of votes.

\[
\{; 0\}; \{\text{So}; 5\}; \{\text{Jr}; 3\}; \{\text{Sr}; 3\}; \{\text{So, Jr}; 8\}; \{\text{So, Sr}; 8\}; \{\text{Jr, Sr}; 6\}; \{\text{So, Jr, Sr}; 11\}
\]

For example, \{\text{Jr, Sr}; 6\} indicates that the junior and senior representatives vote for an issue and that they have a total of six votes between them.

**Mathematician of Note**

John Banzhaf III (1940– )

John Banzhaf, a law professor who also holds an engineering degree, is a well-known consumer rights advocate.
Each of these collections of voters is called a coalition. Those with enough votes to pass an issue are winning coalitions. The winning coalitions in this example are those with six or more votes and are listed below along with their respective vote totals.

\{\text{So, Jr; 8}\} \{\text{So, Sr; 8}\} \{\text{Jr, Sr; 6}\} \{\text{So, Jr, Sr; 11}\}

The last winning coalition is different from the other three in one important respect: If any one of the members decides to vote differently, the coalition still wins. No single member is essential to the coalition. Banzhaf argued that the only time a voter has power is when the voter belongs to a coalition that needs the voter in order to pass an issue. The coalitions for which at least one member is essential are

\{\text{So, Jr; 8}\} \{\text{So, Sr; 8}\} \{\text{Jr, Sr; 6}\}.

Notice that the sophomore representative is essential to two of the coalitions, which is also true of the junior and senior representatives. In other words, about the same number of times, each of the representatives can be expected to cast a key vote in passing an issue.

A paradox: Although the votes have been distributed to give greater power to the sophomores, the outcome is that all members have the same power!

Since distributing the votes in a way that reflects the population distribution does not always result in a fair distribution of power, mathematical procedures can be used to develop ways to measure actual power in weighted voting situations.

A measure of the power of a member of a voting body is called a power index. In this lesson, a voter’s power index is the number of winning coalitions in which the voter is essential. For example, in the student council situation, the sophomore representative is essential to two winning coalitions and thereby has a power index of 2, as do the junior and senior representatives.
Lesson 1.5  • Weighted Voting and Voting Power

A Power Index Algorithm

1. List all coalitions of voters that are winning coalitions.
2. Select any voter, and record a 0 for that voter’s power index.
3. From the list in step 1, select a coalition of which the voter selected in step 2 is a member. Subtract the number of votes the voter has from the coalition’s total. If the result is less than the number of votes required to pass an issue, add 1 to the voter’s power index.
4. Repeat step 3 until all coalitions of which the voter chosen in step 2 is a member are checked.
5. Repeat steps 2 through 4 until all voters are checked.

Exercises

1. Consider a situation in which A, B, and C have 3, 2, and 1 votes respectively, and in which 4 votes are required to pass an issue.
   a. List all possible coalitions and all winning coalitions.
   b. Determine a power index for each voter.
   c. If the number of votes required to pass an issue is increased from 4 to 5, determine a power index for each voter.

2. In a situation with three voters, A has 7 votes, B has 3, and C has 3. A simple majority is required to pass an issue.
   a. Determine a power index for each voter.
   b. A dictator is a member of a voting body who has all the power. A dummy is a member who has no power. Are there any dictators or dummies in this situation?

3. The student council example in this lesson depicts a situation with three voters that results in equal power for all three. In Exercises 1 and 2, power is distributed differently. Find a distribution of votes that results in a power distribution among three voters that is different from the ones you have already seen. How many new power distributions in situations with three voters can you find?
4. In this lesson’s student council example, can the votes be distributed so that the members’ power indices are proportionate to the class sizes? Explain.

5. In this lesson’s student council example, suppose that the representatives of the junior and senior classes always differ on issues and never vote alike. Does this make any practical difference in the power of the three representatives? Explain.

6. (See Exercise 14 of Lesson 1.4 on page 33.) Let \( C_n \) represent the number of coalitions that can be formed in a group of \( n \) voters. Write a recurrence relation that describes the relationship between \( C_n \) and \( C_{n-1} \).

7. One way to determine all winning coalitions in a weighted voting situation is to work from a list of all possible coalitions. Use A, B, C, and D to represent the individuals in a group of four voters and list all possible coalitions.

8. Weighted voting is commonly used to decide issues at meetings of corporate stockholders. Each member has one vote for each share of stock held.
   a. A company has four stockholders: A, B, C, and D. They own 26%, 25%, 25%, and 24% of the stock, respectively, and more than 50% of the vote is needed to pass an issue. Determine a power index for each stockholder. (Use your results from Exercise 7 as an aid.)
   b. Another company has four stockholders. They own 47%, 41%, 7%, and 5% of the stock. Find a power index for each stockholder.
   c. Compare the percentage of stock owned by the smallest shareholder in parts a and b. Do the same for the power index of the smallest stockholder in each case.
9. A landmark court decision on voting power involved the Nassau County, New York, Board of Supervisors. In 1964, the board had six members. The number of votes given to each was 31, 31, 21, 28, 2, and 2.

a. Determine a power index for each member.

b. The board was composed of representatives of five municipalities with these populations:

<table>
<thead>
<tr>
<th>Municipality</th>
<th>Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hempstead</td>
<td>728,625</td>
</tr>
<tr>
<td>North Hempstead</td>
<td>213,225</td>
</tr>
<tr>
<td>Oyster Bay</td>
<td>285,545</td>
</tr>
<tr>
<td>Glen Cove</td>
<td>22,752</td>
</tr>
<tr>
<td>Long Beach</td>
<td>25,654</td>
</tr>
</tbody>
</table>

The members with 31 votes represented Hempstead. The others each represented the municipality listed in the same order as in the table. Compare the power indices of the municipalities with their populations.

10. A minimal winning coalition is one in which all voters are essential.

a. Give an example of a weighted voting situation with a winning coalition for which at least one but not all of the voters is essential. Identify the minimal winning coalitions in this situation.

b. Is defining a voter’s power index as the number of minimal winning coalitions to which the voter belongs equivalent to the definition used in this lesson? Explain.
11. The president of the United States is chosen in the Electoral College, a system that can be considered a form of weighted voting among the states. The number of electors given to each state (and the District of Columbia) is equal to its representation in Congress. That is, the number of electors equals the number of members of the House of Representatives plus two (the number of senators). In 2000, Albert Gore won the popular vote by about half a million votes over George Bush, but lost in the Electoral College.

a. In the 2000 census, the population of California (the most populous state) was 33,871,648. The population of Wyoming (the least populous state) was 493,782. California has 52 representatives in the U.S. House. Wyoming has 1. Use these data to construct an argument that the electoral vote distribution is weighted in favor of small states.

b. Some reformers have proposed removing the electoral weighting. They suggest equating the number of electors with the membership in the U.S. House only. The following table shows the Electoral College votes in the 2000 election. Would this reform proposal change the 2000 election results? Explain.

<table>
<thead>
<tr>
<th>State</th>
<th>Bush</th>
<th>Gore</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alabama</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>Alaska</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Arizona</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>Arkansas</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>California</td>
<td>54</td>
<td></td>
</tr>
<tr>
<td>Colorado</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>Connecticut</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>Delaware</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>D. C.</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Florida</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>Georgia</td>
<td>13</td>
<td></td>
</tr>
<tr>
<td>Hawaii</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Idaho</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Illinois</td>
<td>22</td>
<td></td>
</tr>
<tr>
<td>Indiana</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>Iowa</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>Kansas</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>Kentucky</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>Louisiana</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>Maine</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Maryland</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>Massachusetts</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>Michigan</td>
<td>18</td>
<td></td>
</tr>
<tr>
<td>Minnesota</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>Mississippi</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>Missouri</td>
<td>11</td>
<td></td>
</tr>
<tr>
<td>Montana</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Nebraska</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Nevada</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>New Hampshire</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>New Jersey</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>New Mexico</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>New York</td>
<td>33</td>
<td></td>
</tr>
<tr>
<td>North Carolina</td>
<td>14</td>
<td></td>
</tr>
<tr>
<td>North Dakota</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Ohio</td>
<td>21</td>
<td></td>
</tr>
<tr>
<td>Oklahoma</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>Oregon</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>Pennsylvania</td>
<td>23</td>
<td></td>
</tr>
<tr>
<td>Rhode Island</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>South Carolina</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>South Dakota</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Tennessee</td>
<td>11</td>
<td></td>
</tr>
<tr>
<td>Texas</td>
<td>32</td>
<td></td>
</tr>
<tr>
<td>Utah</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Vermont</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Virginia</td>
<td>13</td>
<td></td>
</tr>
<tr>
<td>Washington</td>
<td>11</td>
<td></td>
</tr>
<tr>
<td>West Virginia</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Wisconsin</td>
<td>11</td>
<td></td>
</tr>
<tr>
<td>Wyoming</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>
c. The Electoral College is mandated in the United States Constitution. Why do you think the founders of the country instituted a system weighted in favor of small states? (Do some research if you are not sure.)

**Computer Explorations**

12. Use the weighted voting software that accompanies this book to experiment with different weighted voting systems when there are three voters. Change the number of votes given to each voter and the number of votes required to pass an issue. How many different power distributions are possible? Do the same for weighted voting systems with four voters.

**Projects**

13. The Security Council of the United Nations is composed of five permanent members and ten others who are elected to two-year terms. For a measure to pass, it must receive at least nine votes that include all five of the permanent members. Determine a power index for a permanent member and for a temporary member. (Assume that all members are present and voting.)

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**Majority of Member States Back Pesticide Ban**

*European Voice*  
April 29, 2013

A majority of member states today voted to approve a European Commission proposal to ban the use of neonicotinoid seed treatment pesticides, thought to be harmful to bees.

The outcome was close, following a first vote last month which was inconclusive, with neither a majority for or against. Crucially, Germany changed its position in today’s vote to approve the ban, after having abstained last month. The 15 member states voting in favor gave 189 weighted votes, versus 125 weighted votes against, including those of the UK. Abstentions amounted to 33 weighted votes.
14. Research and report on other power indices. What, for example, is the Shapley-Shubik power index?

15. What is the effect of the Electoral College system on the power of individual voters in selecting the president? Research the matter and report on the relative power of voters in different states.

16. Research and report on court decisions about weighted voting.

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**U.K., U.S. Wield Most Cyber Power**

_European Voice_  
April 29, 2013

A new poll released today by Fair Vote Canada shows a majority of Canadians, of all political stripes, favor changing our voting system to incorporate proportional representation.

That would mean instead of just being elected based on winning the most votes, at least some MPs would be elected based on the share their party gets of the popular vote. Canada’s first- past-the-post system is one of the last of its kind in the world, with most other developed nations using some form of proportional representation.

The poll found 70 per cent of Canadians strongly (24 per cent) or somewhat (46 per cent) support moving towards proportional representation.

Just as a reminder of why there is a push to amend our system:

In the 2011 federal election, the Conservatives won 53.8 per cent of the seats and a majority government with few checks on its power, with just 39.6 per cent of the votes. Much the same as the Liberals in 2000 won 57 per cent of the seats with 40.8 per cent of the vote.
Chapter Extension

Proportional Representation

Democracies are founded on the principle that all people should have representation in government. Most democratic countries have minority populations who feel that they should be represented by one of their own members, and courts have agreed.

However, ensuring minority representation in a legislative body such as the United States House of Representatives is not always easy. If, for example, a state has five representatives in the U.S. House and a minority is 40% of the state’s population, it seems reasonable that the minority should hold $0.4 \times 5 = 2$ of the seats. But, depending on how the boundaries of the state’s five congressional districts are drawn, the minority might hold no seats.

The task of ensuring minority representation in the U.S. House must be undertaken every ten years when district boundaries are redrawn after a census. In states with a significant minority population, districts are established in which the minority has over half the population. However, this practice sometimes produces districts with a shape so unusual that courts reject them.

How, then, does a democracy provide fair representation in government? Many democracies use some form of proportional representation. Although there are several proportional representation models in use, they all have a common goal: to ensure that minorities and/or political parties have representation in government proportionate to their numbers in the general population.
One form of proportional representation is achieved through a practice called *cumulative voting*. In this model, several representatives are elected from a single region (i.e., state). For example, if the region has three representatives, each voter has three votes. The voter can split the three votes in any way, and can even cast more than one vote for a single candidate. Cumulative voting is used in some local elections in the United States.

Another model, the *party-list model*, is used in some European countries. In this model, each party has a list of candidates on the ballot. Each voter votes for one of the parties. When the election is over, the party receives a number of seats proportionate to the vote it received. The seats are usually assigned to names on the party’s ballot by taking them in order from the top of the ballot until the correct number is obtained.

The *mixed member model* has voters vote for a party and a candidate. A portion of the seats is assigned to candidates and another portion to parties. All individual winning candidates receive seats. The remaining seats are awarded to members of parties that do not have a number of individual seats proportionate to the vote they received. In 1994, New Zealand voters abandoned a plurality model like the one currently used in the United States in favor of the mixed member model, which is also used in several European countries.

In the *preference vote model*, voters rank the candidates. A threshold is established, and all candidates with a vote total over the threshold are elected. Remaining seats are distributed by conducting a form of sequential runoff among the remaining candidates.
A new poll released today by Fair Vote Canada shows a majority of Canadians, of all political stripes, favor changing our voting system to incorporate proportional representation.

That would mean instead of just being elected based on winning the most votes, at least some MPs would be elected based on the share their party gets of the popular vote. Canada's first-past-the-post system is one of the last of its kind in the world, with most other developed nations using some form of proportional representation.

The poll found 70 per cent of Canadians strongly (24 per cent) or somewhat (46 per cent) support moving towards proportional representation.

Just as a reminder of why there is a push to amend our system: In the 2011 federal election, the Conservatives won 53.8 per cent of the seats and a majority government with few checks on its power, with just 39.6 per cent of the votes. Much the same as the Liberals in 2000 won 57 per cent of the seats with 40.8 per cent of the vote.
Chapter 1 Review

1. Write a summary of what you think are the important points of this chapter.

2. Consider the following set of preferences.
   a. Determine a winner using a 5-4-3-2-1 Borda count.
      
      A  B  C  D  E
      20 22 12 12 9

   b. Determine a plurality winner.
   c. Determine a runoff winner.
   d. Determine a sequential runoff winner.
   e. Determine a Condorcet winner.
   f. Suppose that this election is conducted by an approval model and all voters approve of the first two choices on their preference schedules. Determine an approval winner.

3. Complete the following table for the recurrence relation $B_n = 2B_{n-1} + n$.

<table>
<thead>
<tr>
<th>n</th>
<th>$B_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>2(3) + 2 = 8</td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>
4. In this chapter, you encountered several paradoxes involving group-ranking models.

   a. One of the most surprising paradoxes occurs when a winning choice becomes a loser when the choice's standing actually improves. In which group-ranking model(s) can this occur?

   b. Discuss at least one other paradox that occurs with group-ranking models.

5. In the 1912 presidential election, polls showed that the preferences of voters were as follows.

<table>
<thead>
<tr>
<th></th>
<th>Wilson</th>
<th>Roosevelt</th>
<th>Taft</th>
</tr>
</thead>
<tbody>
<tr>
<td>Roosevelt</td>
<td>45%</td>
<td>30%</td>
<td>25%</td>
</tr>
<tr>
<td>Taft</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Taft</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

   a. Who won the election? Was he a majority winner?

   b. How did the majority of voters feel about the winner?

   c. How could one of the groups of voters have changed the results of the election by voting insincerely?

   d. Discuss who might have won the election if a different model had been used.

6. Your class is ranking soft drinks and someone suggests that the names of the soft drinks be placed in a hat and the group ranking be determined by drawing them from the hat. Which of Arrow's conditions does this method violate?

7. After their final round of skating in the 1995 World Figure Skating Championship, Chen Lu of China, Nicole Bobek of the United States, and Surya Bonaly of France were in first, second, and third place, respectively, with little chance of any remaining skater passing them. However, when American Michelle Kwan skated, she did well enough to move into fourth place. But something else quite surprising happened. Kwan's scores reversed the positions of Bobek and Bonaly. Which of Arrow's conditions did the scoring system violate in this case? Explain.
8. State Arrow’s theorem. In other words, what did Arrow prove?

9. Can the point system used to do a Borda count affect the ranking (for example, a 5-3-2-1 system instead of a 4-3-2-1 system)? Construct an example to support your answer.

10. The 1992 presidential election was unusual because of a strong third-party candidate. In that election Bill Clinton received 43% of the popular vote, George Bush 38%, and Ross Perot 19%.

Steven Brams and Samuel Merrill III used polling results to estimate the percentage of those voting for one candidate who also approved of another.

- Approximately 15% of Clinton voters approved of Bush and approximately 30% approved of Perot.
- Approximately 20% of Bush voters approved of Clinton and approximately 20% approved of Perot.
- Approximately 35% of Perot voters approved of Clinton and approximately 30% approved of Bush.

a. Estimate the percentage of approval votes each candidate would have received if approval voting had been used in the election.

b. Find the total of the three percentages you gave as answers in part a. Explain why the total is not 100%.

11. Choose an election model from those you have studied in this chapter that you think best to use to determine a winner for the following preferences. Explain why you think your choice of method is best.

```
A  D  C  B
B  C  D  C
C  B  B  D
D  A  A  A
32 28 20 10
```
12. In 2012, Washington Nationals outfielder Bryce Harper won the National League Jackie Robinson Rookie of the Year Award. (There are two winners each year: one in the National League and one in the American League.) The results of the National League Voting are shown in the following table.

<table>
<thead>
<tr>
<th>Player</th>
<th>First-Place Votes</th>
<th>Second-Place Votes</th>
<th>Third-Place Votes</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bryce Harper</td>
<td>16</td>
<td>8</td>
<td>8</td>
<td>112</td>
</tr>
<tr>
<td>Wade Miley</td>
<td>12</td>
<td>13</td>
<td>6</td>
<td>105</td>
</tr>
<tr>
<td>Todd Frazier</td>
<td>3</td>
<td>7</td>
<td>9</td>
<td>45</td>
</tr>
<tr>
<td>Wilin Rosario</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>Norichika Aoki</td>
<td>2</td>
<td>5</td>
<td></td>
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<td>Jordan Pacheco</td>
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What type of voting model is used to select rookie of the year?

13. There are conditions other than Arrow’s that some experts consider important to a fair group-ranking model. For example, Donald Saari thinks that a good method should not retain the same winner if the voters reverse their preferences.

a. To see how this reversal effect works, find a plurality winner for the following set of voter preferences. Then reverse the order of the rankings on each schedule and find a plurality winner.

Mathematician of Note

Donald Saari (1940– )
Professor Saari is Distinguished Professor of Mathematics and Economics at University of California Irvine.
b. Do any other models that you studied in this chapter demonstrate a reversal effect in the set of preferences in part a? That is, do any other models leave the winner unchanged when preferences are reversed? Explain.

c. Examine the following set of preferences for reversal effects.

14. Consider a situation in which voters A, B, C, and D have 4, 3, 3, and 2 votes, respectively, and 7 votes are needed to pass an issue.

a. List all winning coalitions and their vote totals.

b. Find a power index for each voter.

c. Do the power indices reflect the distribution of votes? Explain.

d. Suppose the number of votes necessary to pass an issue increases from 7 to 8. How does this change the voters’ power indices?

15. A county planning commission has five members. Each member’s vote is weighted to reflect the population of the community the member represents. Member A has 1 vote, B has 1 vote, C has 2 votes, D has 5 votes, and E has 6 votes. A simple majority of the vote total is required to pass an issue. Do any of the members seem to have considerably more or less power than intended? Explain.
Chapter 1 • Review

Bibliography


