The Mathematics of Conflict

HiMAP Module 3

HIGH SCHOOL MATHEMATICS AND ITS APPLICATIONS (HiMAP) PROJECT

The goal of HiMAP is to develop, through a community of users and developers, a system of instructional modules in high school mathematics and its applications that may be used to enhance teacher training at the secondary school level. The Project is guided by a national advisory board and an editorial board of mathematicians, scientists, and educators. HiMAP is funded by the National Science Foundation to the Consortium for Mathematics and Its Applications (COMAP), Inc., a non-profit corporation engaged in research and curriculum development in mathematics education.

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Preface

The purpose of this module is to demonstrate the usefulness of rigorous mathematical models for analyzing conflict-of-interest situations. By demonstrating that several mathematical techniques are actually applicable to real-world problems and situations, student appreciation of and interest in mathematics should be enhanced. The specific topics included in this module include simple factoring, work with fractions, and elementary probability theory. Many of the principle concepts of the mathematical theory of games, such as the distinction between zero-sum and nonzero-sum games, the notion of an equilibrium outcome, and the pathologies of "Prisoners' Dilemma" games, are also introduced and illustrated with relevant examples.

The module includes several features that should enhance your ability to introduce this topic in the classroom. First, twenty-eight exercises, with answers, are included in the body of the module. The exercises can be used as the basis for classroom discussion or as homework or other assignments. Several teaching lessons, with transparencies, are also provided. They can be used independently, or in conjunction with the exercises. Many of the examples given in the module itself can also be used to motivate student interest and to provide compelling problems for which the mathematical topics mentioned above provide solutions.

How to Use This Module

This module is designed so that the main text appears in large type on the inside, wider columns of the pages. The Lesson Plans material appears in the tinted sidebars. The text for the Lesson Plans is juxtapositioned to its appropriate text in the main article.

Some of the graphic devices used to aid the teachers in the Lesson Plans are as follows:

1. Pages 1 through 28 are devoted to the main module and the Lesson Plans material in the format mentioned above. The subsequent 10 pages contain worksheets, worksheet answers, and transparency masters. These pages are perforated so they may be removed easily for photocopying and for use in the classroom.

2. For ease of reference, a reduced form of each transparency is positioned at the appropriate place in the Lesson Plans.

The Mathematics of Conflict

by

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This material was prepared with the support of the National Science Foundation (NSF) Grant No. DPE8318104. Recommendations expressed are those of the author and do not necessarily reflect the views of the NSF or of the copyright holder.
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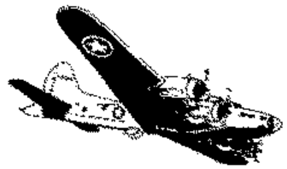
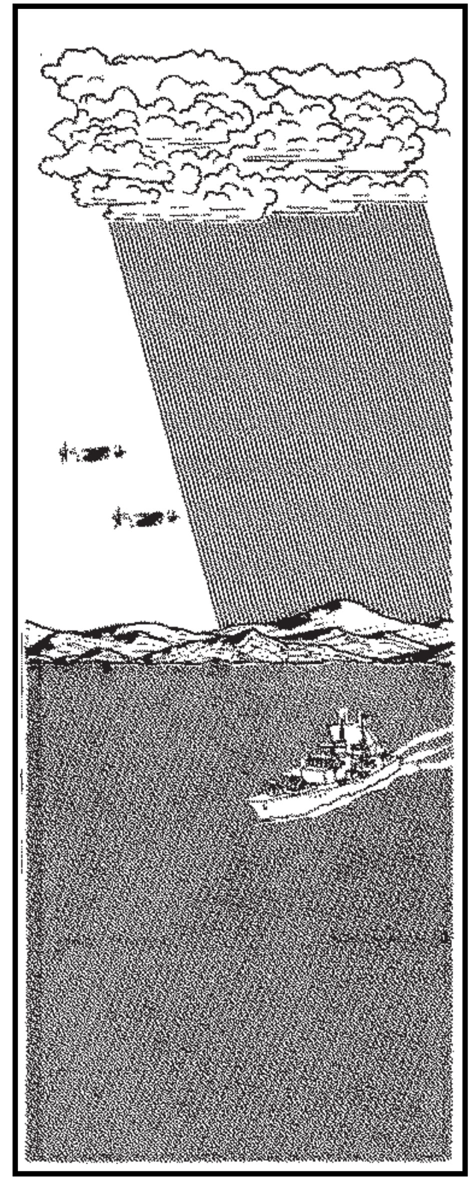
Published by: COMAP, Inc., 57 Bedford Street, Suite 210, Lexington, MA 02420

ISBN 0889-2652

Section 1. Introduction

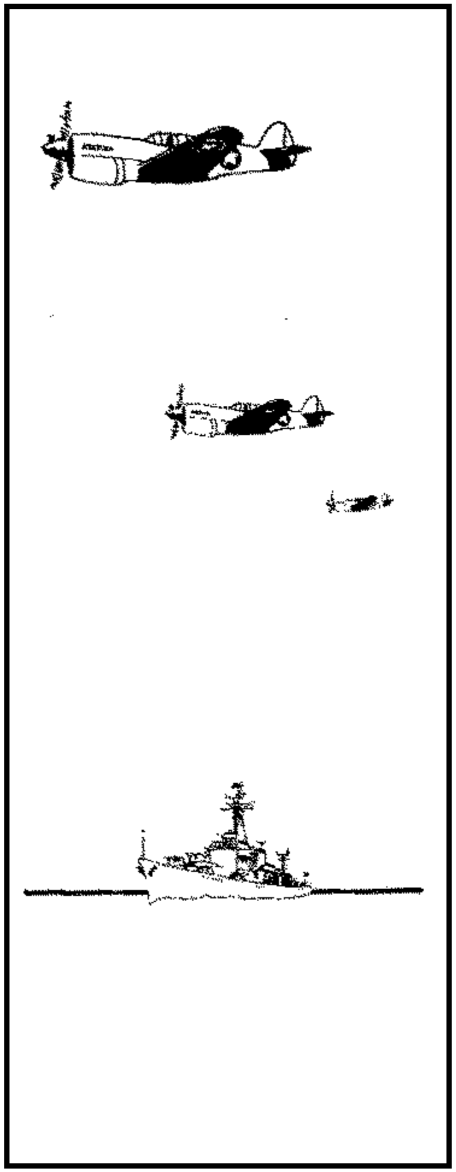




In February 1943, at a critical stage of the struggle for New Guinea during World War II, the Japanese decided to bring reinforcements from the nearby island of New Britain. In moving their troops, the Japanese could either route their transport ships around the northern part of New Britain, where rain and poor visibility were expected, or around the southern tip of the island, where clear weather was expected. In either case, the trip would take three days. Which route should the Japanese take?

If the Japanese were interested in minimizing the number of days that their transport ships would be at sea, they clearly would be indifferent between the two possible routes. And if they wanted their troops to enjoy the weather, they would obviously have selected the southern route. But this was war and the Japanese military command wanted to *minimize* the number of days their transport ships would be exposed to attack by American forces under the command of General George Kenney, the Commander of the Allied Air Force in the Southwest Pacific. Under these conditions, which route should the Japanese take?

Like the Japanese, General Kenney also faced a difficult choice at this time. Allied intelligence had detected evidence of the Japanese convoy assembling at the far side of New Britain. Kenney, of course, wanted to *maximize* the number of days his bombers would be able to attack the convoy but he didn't have enough reconnaissance aircraft to saturate both the northern and southern routes and, hence, ensure early detection of the Japanese transport ships. Thus, Kenney could have concentrated the bulk of his reconnaissance aircraft along either the northern or southern routes. What should Kenney do?



Before answering these questions, notice that the goals of the Japanese command and of General Kenney were interconnected or interdependent. Thus, Kenney's choice of a strategy would have an impact on the ability of the Japanese to implement their objectives, which the choice of the Japanese would also affect the amount of time Kenney would have to bomb the Japanese convoy. Interdependent choice situations like this one are called *games*; and the mathematical *theory of games* can be used to provide an answer to the questions we have posed.

The interdependent nature of the choices of the two players is more apparent when the *outcome matrix* in Figure 1 is examined. In this figure, the two strategies of General Kenney—either to search north or to search south—have been assigned to the columns. The numbers in each cell of the matrix are estimates provided General Kenney by his intelligence staff of the number of days he would have to bomb the Japanese convoy given the selection of each of the logically possible combination of strategies. In this case, since each player had just two strategies, there are 2 x 2 = 4 possible outcomes. For example, if General Kenney concentrated his reconnaissance aircraft along the southern part of New Britain, and the Japanese command had decided to route their transport ships along the southern route, American intelligence estimated that the Japanese convoy would be spotted almost as soon as it left port; hence, Kenney's bombers would have almost three full days to attack it. Note that if the Japanese sail north, Kenney's best strategy is to search north, but if the Japanese sail south, Kenney's best strategy is to search south. In other words, what Kenney should do depends upon what the Japanese do, and vice versa.

Also note that the numbers in each cell of the payoff matrix represent the payoff to the row player, here, General Kenney. A second number could have been added to represent the payoff to the column player, i.e., the Japanese, but in this case the second number would be redundant. This is because the payoff to the Japanese is exactly the opposite of the payoff to General Kenney. For example, if General Kenney's payoff is +3, representing three days of bombing, the Japanese payoff is –3, or three days of being bombed. Games like this one, in which the payoffs to the players are diametrically opposed, are called *zero-sum games.* The reason for the name is that the sum of the payoffs for any one outcome sum to zero, that is (3) + (–3) = 0.

Japanese Commander

		Sail North	Sail South	Minimum of Rows
General Kenney	Search North	2 days	2 days	②
	Search South	1 day	3 days	1
Maximum of Columns		②	3	

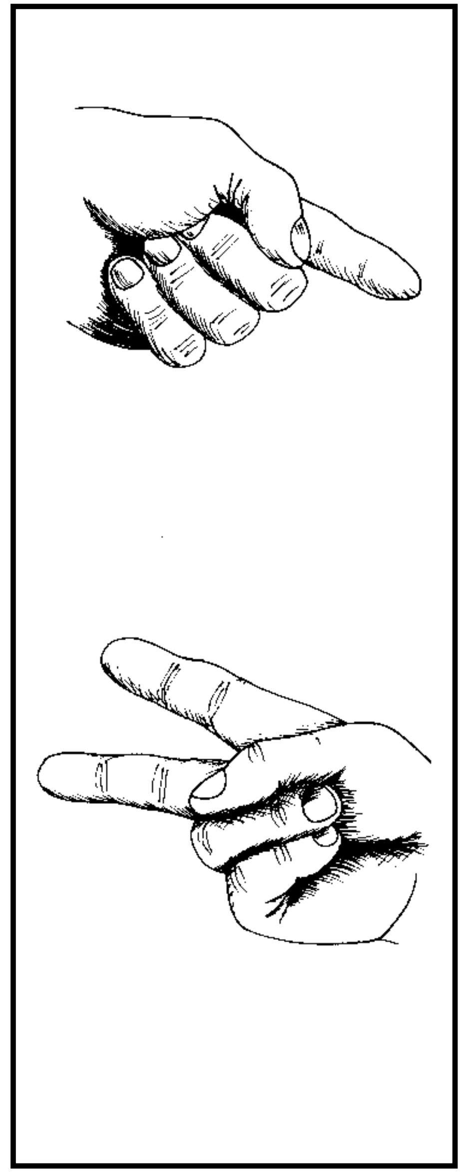
Figure 1

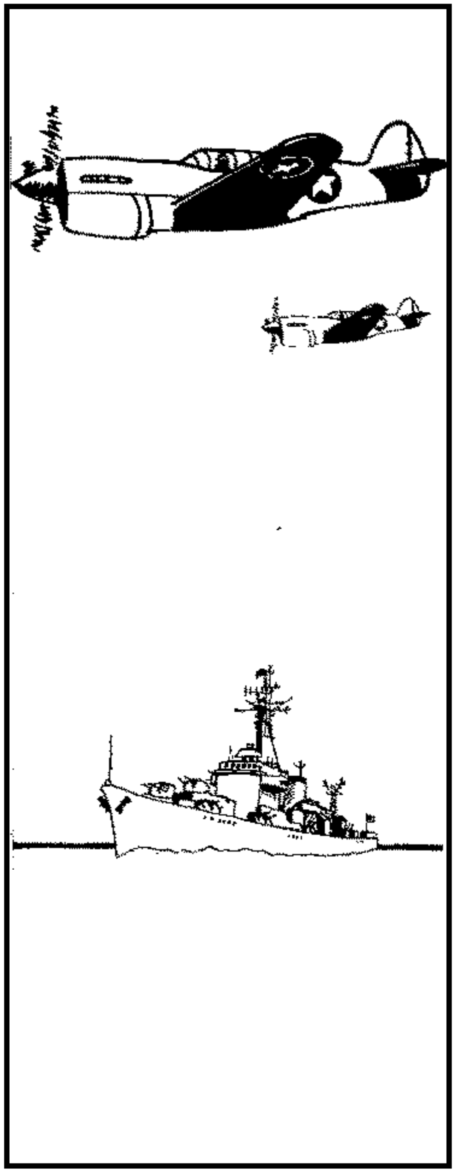
Exercise 1

Construct an outcome matrix for the game of odds and evens. Assume that each player simultaneously puts out one or two fingers. If the number of fingers sums to an even integer, the row player (Mr. Even) wins $5.00 from the column player (Mr. Odd). If the number of fingers sum to an odd number, the column player wins $5.00 from the row player.

Exercise 2

Construct an outcome matrix for the game of odds and evens but this time assume that Mr. Even can put out one, two, or three fingers. How many outcomes are possible if both players are able to use three fingers?





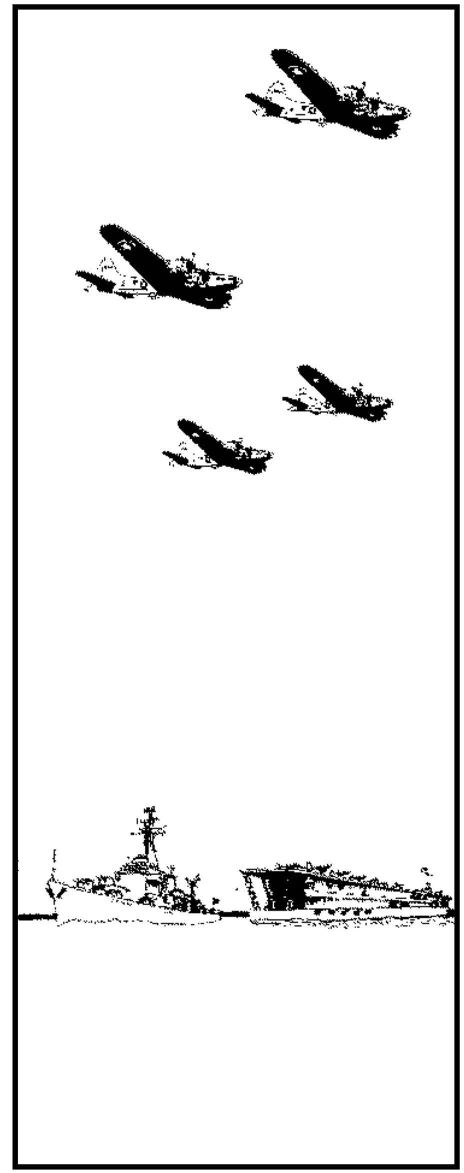
Section 2. Identifying Optimal Strategies

On what basis should players in zero-sum games like this one choose strategies?Although there are many strategic principles that could be evoked, game theorists are in general agreement that the *minimax* (or *maximin*) principle, which essentially requires that players maximize their *security level* (to be defined presently) is the soundest.

To determine each player's best strategy, then, notice first that if General Kenney chooses his search north strategy, the worst outcome that could result is two days of bombing, while the worst possible outcome for Kenney, should he choose to search south, is one day of bombing. Define the worst outcome associated with each strategy as the security level of that strategy. The security level of each of General Kenney's two strategies—that is, the minimum entry of each row–is indicated to the right of the outcome matrix of Figure 1.

It is easy to see that the security level of General Kenney's search north strategy (i.e., 2) is higher than the security level of his search south strategy (i.e., 1). Thus, to maximize his *security level*, Kenney should search north. Since this is associated with the outcome that is the *maximum* of the *minimum* entry in each row, it is called the *maximin* strategy.

The security level for each strategy of the Japanese command is listed just below Figure 1. Recall that the payoffs of the two players in this game are diametrically opposed. Thus, the higher the payoff to the row player, the lower the payoff to the column player. For instance, the worst outcome associated with the sail north strategy of the Japanese is two days of bombing while the security level of the Japanese sail south strategy is three. Therefore, in order to maximize their security level, the Japanese should sail north. Since this choice entails choosing a strategy associated with the *minimum* of the column *maxima*, it is called a *minimax* strategy.

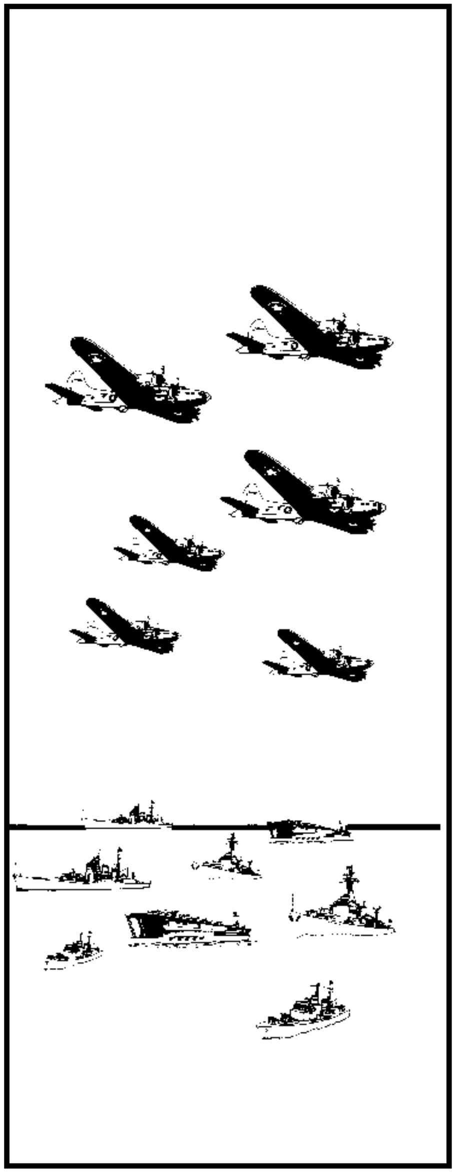


Notice that the maximin and minimax strategies of the two players are associated with the same outcome (i.e., the value 2, which is circled) in Figure 1. When this happens, the maximin and minimax strategies are said to be in equilibrium, and the outcome associated with them is called an *equilibrium outcome* (or *saddlepoint*). When two strategies are in equilibrium, neither player has an incentive unilaterally to change his strategy. For instance, should General Kenney switch from his search north strategy to his search south strategy, he would reduce from two to one the number of days he would have to bomb the Japanese convoy. Likewise, the Japanese would have no incentive to switch from their sail north to their sail south strategy. In either case, their convoy would be bombed for two days.

Exercise 3
Explain why payoff "3" is not an equilibrium outcome.

Exercise 4
Determine whether any other outcome in this game is an equilibrium outcome.

In zero-sum games with an equilibrium outcome or saddlepoint, strategies associated with a saddlepoint are considered *optimal* and players who choose them are termed *rational*. There are a number of good reasons for this. First, equilibrium strategies secure for each player the *value* of the game, that is, the best outcome that either player can assure himself of against a rational opponent. (In a game with a saddlepoint, the value of the game is always equal to the saddlepoint.) Second, these strategies maximize each player's security level. Finally, equilibrium strategies, when they exist, are the best response to the strategy that maximized the other player's security level. For example, the best response of General Kenney to the Japanese minimax strategy is the strategy that maximizes his own security level. This implies that in games with a saddlepoint, if a player has advance information that his opponent plans to choose his optimal strategy, he is not helped with this knowledge, nor is his opponent hurt. In a way, then, zero-sum games with a saddlepoint are *strictly determined*, and they are sometimes referred to by this label.



As a historical footnote, both players in this game chose their maximin and the minimax strategies associated with the unique equilibrium outcome in this game, and thereby maximized their respective security levels. General Kenney concentrated the bulk of his reconnaissance aircraft along the northern part of New Britain, while the Japanese commander decided to route his convoy along the northern route. Consequently, the Japanese convoy was subject to two days of bombing. Unbeknownst to the Japanese, however, Kenney had modified a number of his aircraft for low-level bombing,, thereby inflicting heavy losses on the Japanese convoy.

Exercise 5

Explain why O. G. Haywood [2], in using game theory to analyze this battle maintains that one could not say that "the Japanese commander erred in his decision," even in the light of the disastrous losses to the convoy. In what sense can one disagree with Haywood?

Exercise 6

Assume that Allied intelligence had determined that they would have two days to bomb the Japanese convoy if the Japanese had sailed north and Kenney searched south. Construct a new outcome matrix and identify the rational strategies for the players under these conditions.

Exercise 7

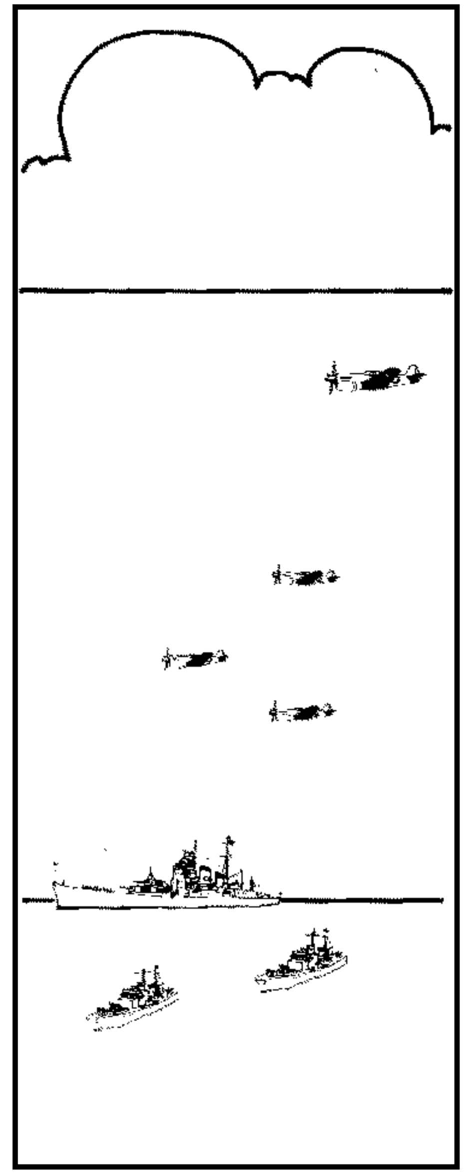
Verify that saddlepoints uniquely correspond to those entries of the outcome matrix which are simultaneously the minimum entry of its row and the maximum entry of its column.

Section 3. Zero-Sum Games Without a Saddlepoint

Consider now another situation similar to the game played between the Japanese command and General Kenney, but with the payoffs as listed in Figure 2. Assume that the two players in this game will face a number of situations exactly like this one in the next few weeks, that is, that this game will be repeated several times.

		General B: Sail North	General B: Sail South	Row Minimum
General A	Search North	1 day	5 days	1
	Search South	6 days	4 days	(4)
	Column Maximum	6	(5)	

Figure 2



In this hypothetical game, neither player has an equilibrium strategy since the strategies that minimize each player's security level—search south for General *A* and sail south for General *B*—are *not* associated with the same value. General *A*'s security level is associated with the payoff "4" while General *B*'s security level is associated with the payoff "5". Also note that there is no single entry in the figure that is simultaneously the minimum entry of its row and the maximum entry of its column. Thus, this is an example of a zero-sum game *without* a saddlepoint. Neither player in this game, therefore, possesses an equilibrium strategy.

Lesson 1. Games

The purpose of this lesson is to provide students with a sense of the concept of a game. An interesting way to do this is to contrast a "game" with a "decision." Decisions are commonly understood to involve choices that lead to outcomes that are independent of the actions of other actors. The simplest decisions are made under conditions of certainty, that is, when the consequences of each possible choice are known in advance. For example, in an elevator, the consequences of each choice are indicated by the buttons marked with numbers indicating floor stops. A decision is made simply by selecting that button associated with the preferred stop.

Decisions can also be made under conditions of uncertainty, that is, when the consequences of each possible choice are not known in advance. This might occur if the numbers in an elevator are worn away. Under these conditions, it might be difficult to know whether the elevator will top at floor "3" or "8." But even in this case, the eventual outcome does not depend upon the choices of another actor. Students might be asked to discuss the reasons why an elevator might not stop on the desired floor.

That both players in this game lack an equilibrium strategy means that at least one player has an incentive to switch to another strategy at every outcome. For instance, at outcome "6," General B has an incentive to switch strategies in order to reduce from six to four the number of days his troops are exposed to General A's bombers. It also means that if either player in this game knew what strategy his opponent intended to use, he could exploit this information to maximize his payoff. This suggests the strategic principle that a player in this game should select a strategy in such a way as to keep information about his strategy choice from his opponent. Optimal strategies for zero-sum without a saddlepoint are based upon this supposition.

Exercise 8

Verify the fact that there is no equilibrium outcome in the game of Figure 2.

One way for a player to keep information about his strategy choice from his opponent is to select a strategy randomly according to a particular probability distribution. Such a strategy is called a mixed strategy. Because it is random and hence, cannot be predicted, the use of a mixed strategy can prevent one's opponent from discovering in advance what strategy a player intends to use. For instance, General A could decide to search north with relative frequency (or probability) p_1 and to search south with frequency p_2, where $p_1 > 0, p_2 > 0$, and $p_1 + p_2 = 1$. Similarly, General B could decide to sail north with frequency q_1 and to sail south with frequency q_2 where $q_1 \geq 0$, $q_2 \geq 0$, and $q_1 + q_2 = 1$.

In this case, the average or expected payoff for General A, determined by summing each payoff, weighted by the probability that it would occur, would be:

$$E(A) = 1(p_1q_1) + 5(p_1q_2) + 6(p_2q_1) + 4(p_2q_2)$$

Since $p_1 + p_2 = 1$, and $q_1 + q_2 = 1$, $p_2 = (1 - p_1)$ and $q_2 = (1 - q_1)$. By substitution:

$$E(A) = 1(p_1q_1) + 5(p_1)(1 - q_1) + 6(1 - p_1)(q_1) + 4(1 - p_1)(1 - q_1).$$

Multiplying and collecting terms gives:

$$E(A) = -6(p_1 q_1) + p_1 + 2(q_1) + 4$$

To determine the values of p_1 and p_2 that maximize General A's expected payoff, a four-step manipulation process, suggesting in [5] can be used.

(1) First, factor out p_1 from the terms that include it, which gives
$E(A) = p_1(-6q_1 + 1) + 2(q_1) + 4$

(2). Then arrange the q_1 terms to have a coefficient of 1. Hence
$E(A) = -6(p_1)(q_1 - 1/6) + 2(q_1) + 4$

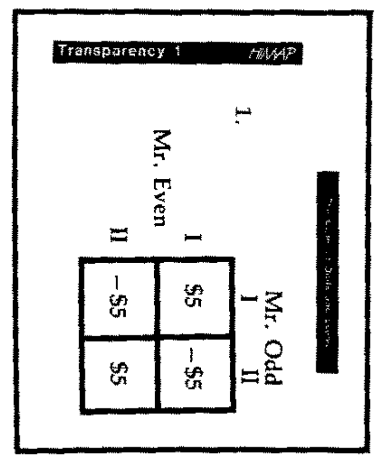
(3) Next, by adding or subtracting a constant, arrange the second q_1 term to be the same as the first:
$E(A) = -6(p_1)(q_1 - 1/6) + 2(q_1 - 1/6) + 4 + (2)(1/6)$

(4) Finally, factor the q terms to manipulate the expression into the form
$K(p_1 - a)(q_1 - b) + c$

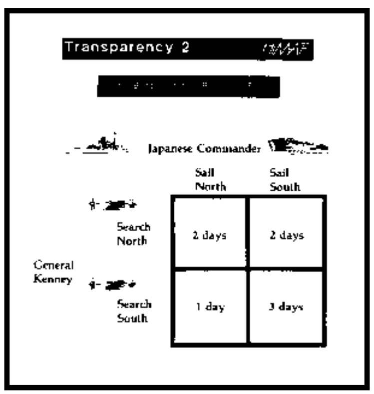
$$E(A) = -6(p_1 - 1/3)(q_1 - 1/6) + 4\ 1/3.$$

With the expression in this form, it can be seen that General A can ensure an expected payoff of 4 1/3 if he chooses to search north with probability $p_1 = 1/3$ (and hence chooses to search south with probability $p_2 = 2/3$), no matter what strategy General B uses, since the first factor of the expression for $E(A)$ is equal to zero and therefore the product is zero when $p_1 = 1/3$. Similarly, General B, whose expected payoff is the negative of General A's, can ensure a payoff of (–4 1/3) if he chooses to sail north with probability $q_1 = 1/6$ and to sail south with probability $q_2 = 5/6$.

To distinguish decisions from games, you can show your class Transparency 1. This is the simple game of ODDS AND EVENS which is described in the module. Show the students that the outcome of this game depends on the choices of both players. Ask them what they would do if they were Mr. Even and they knew that Mr. Odd was going to put out 1 (or 2) finger(s). Demonstrate that when Mr. Odd puts out 1 finger, Mr. Even should put out 1 finger, and when Mr. Odd puts out 2 fingers, Mr. Even should do the same. Point out that this situation is essentially a decision when either player knows in advance what the other is going to do. Also point out, however, that it is a game when neither player knows what the other is going to do.

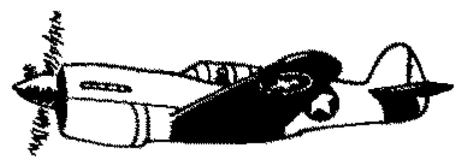


Ask students to think of other situations in which the consequences of a choice depend upon the actions of other actors. Ask them to distinguish games from decisions. Ask them how they would play this game against an opponent who strictly alternated between 1 and 2 fingers. Ask them how they would play this game if they did not know what the other player was going to do. Discuss other zero-sum games given in the module (the game between General Kenney and the Japanese Commander played during the Battle of the Bismarck Sea is given in Transparency 2) and show how games with and without an equilibrium outcome differ. Demonstrate how optimal mixed strategies provide a solution for the game of Odds and Evens.



The *mixed strategy* pairs (1/3, 2/3)—for General *A*—and (1/6, 5/6)—for General *B* are the optimal strategies for the players in this game. By using these strategies, each player in effect raises his security level. Recall that General *A* could ensure a payoff of 4, while General *B* could ensure a payoff of –5, if each were limited to the selection of a single strategy. In addition, the use of these optimal mixed strategies ensures for each player the value of the game which, as in games with a saddlepoint, is the payoff that the row player (here, General *A*) can guarantee himself by selecting an optimal strategy (i.e., 4 1/3).

Since General *A*'s optimal mixed strategy maximizes his minimum gain, it is called a maximin strategy. For similar reasons, General *B*'s optimal mixed strategy is called a minimax strategy. Thus, in both categories of zero-sum games—games with and without a saddlepoint—the strategy that maximized each player's security level is referred to as either a maximin or a minimax strategy.



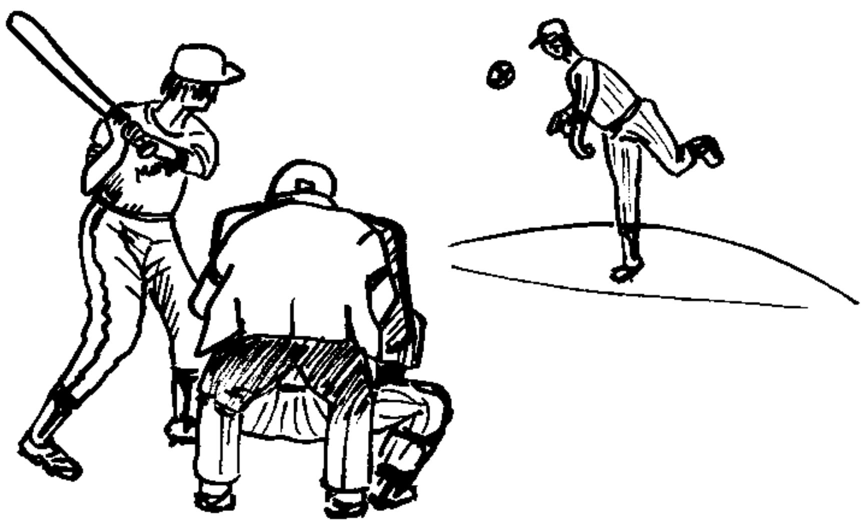
Significantly, in games without a saddlepoint, optimal mixed strategies are in equilibrium. Neither player can increase his expected payoff by switching to another strategy if his opponent does not switch to a non-optimal strategy. In these games, then, just as in games with a saddlepoint, players are not helped, nor are their opponents hurt, if they know in advance that their opponent is going to choose his optimal strategy, that is, the particular probability distribution according to which he will ultimately make a strategy choice. Of course, a player always benefits in these games if he can determine the strategy his opponent will select on any one play of the game. But optimal mixed strategies have the advantage that they prevent one's opponent from obtaining information of this sort.

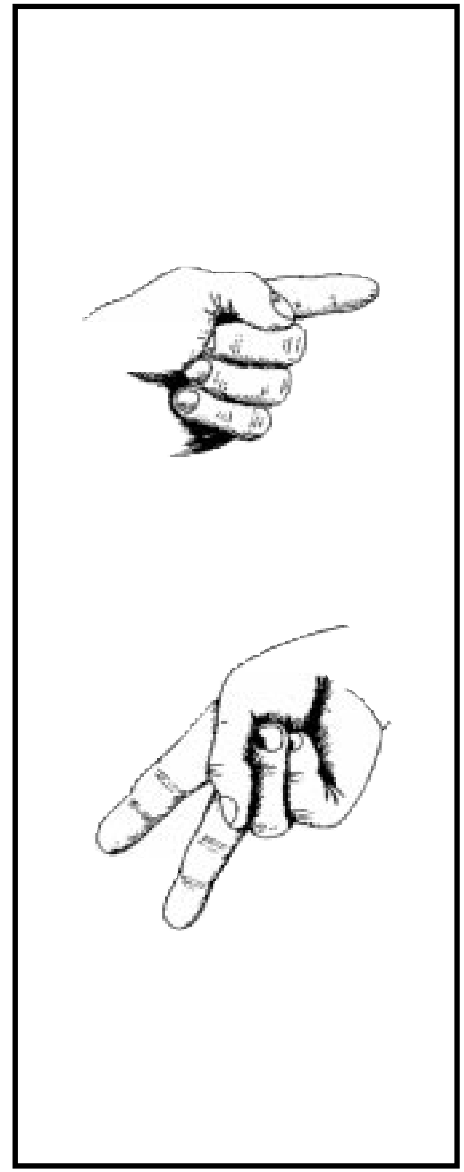
Exercise 9

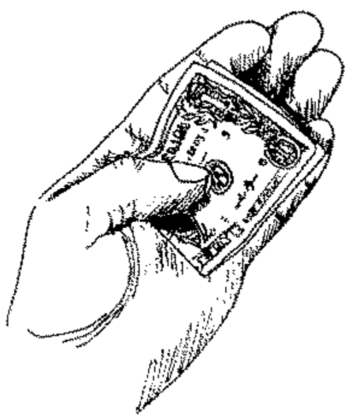
Explain why the confrontation between a pitcher and a batter in baseball can be modeled as a zero-sum game without a saddlepoint. Also explain the relevance of the notion of a mixed strategy in this repeated game situation.

Exercise 10

Using the manipulation procedure outlined above, determine the optimal mixed strategy for each player in the odds and evens game discussed in Exercise 1. Explain why neither player gains an advantage if his opponent knows the optimal probability distribution that a player uses to select a strategy. Explain why either player would benefit from knowing the actual strategy sequence his opponent intends to use.







Section 4. An Alternative Method for Determining Optimal Mixed Strategies

Let us consider now the following zero-sum game without a saddlepoint given in [3]. Mr. Jones conceals either a $1 or a $2 bill. Mr. Smith guesses either $1 or $2 and wins the bill if he guesses correctly. Otherwise, he gets nothing. The following matrix summarizes the strategic situation for the two players:

		Mr. Smith Guesses: $1	$2	Row Minimum
Mr. Jones Hides:	$1	−$1	0	(−$1)
	$2	0	−$2	−$2
Column Maximum		(0)	(0)	

Figure 3.

In this game, since there is no outcome that is simultaneously the column maximum and the row minimum, there is no saddlepoint. Assuming that this game is to be repeated, each player will maximize his security level by choosing his optimal mixed strategy. In 2 x 2 games with the following form

	Player B	
Player A	a	b
	c	d

Figure 4.

the formulas listed below provide an alternate method for determining each player's optimal mixed strategy:

$$p_1 = \frac{d - c}{a + d - b - c} \qquad p_2 = \frac{a - b}{a + d - b - c}$$

$$q_1 = \frac{d - b}{a + d - b - c} \qquad q_2 = \frac{a - c}{a + d - b - c}$$

These formulas can be derived as follows. If the row player uses his optimal mixed strategy, row 1, with probability p_1, and row 2, with probability p_2, then the payoff to row is the same regardless of what mixed strategies column plays. In particular, when row plays his optimal strategy, the payoff to row if column plays column 1 all the time equals the payoff to row if column plays column 2 all the time.

Hence

$$p_1(a) + (1 - p_1)c = p_1 b + (1 - p_1)d.$$

This simplifies to :

$$p_1 = \frac{d - c}{a + d - b - c}$$

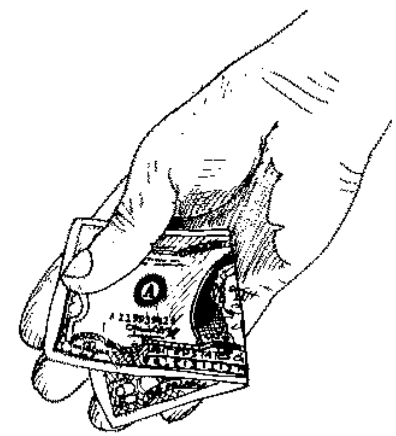
The other formulas are similarly derived. The denominator of each formula is

$$a + d - b - c = (-1) + (-2) - 0 - 0 = (-3)$$

Applying the formulas:

$$p_1 = \frac{-2}{-3} = \frac{2}{3} \qquad p_2 = \frac{-1}{-3} = \frac{1}{3}$$

$$q_1 = \frac{-2}{-3} = \frac{2}{3} \qquad q_2 = \frac{-1}{-3} = \frac{1}{3}$$



Exercise 11

Use these formulas to determine the optimal mixed strategies of the players in the game of Figure 2.

Exercise 12

Use the algebraic procedure outlined in the preceding section to determine the optimal strategies of the players in the game of Figure 3.

In the guessing game between Mr. Smith and Mr. Jones, Mr. Smith clearly has the advantage. He can either win \$1 or \$2, but cannot lose. Mr. Jones, by contrast, can lose either \$1 or \$2. At best, he will break even. How large is the advantage enjoyed by Mr. Smith? Put in a different way, how much would Mr. Smith have to pay Mr. Jones to make this a fair game, that is a game with a value of zero? The following formula for the value, v, of the game—which can also be derived by applying the algebraic method to the matrix of Figure 4—provides the answer to this question:

$$v = \frac{ad - bc}{(a + d) - (b - c)} = \frac{(-1 \times -2) - (0 \times 0)}{-3} = -\frac{2}{3}$$

Since the value of this game is –2/3, Mr. Jones can expect to lose 66 and 2/3 cents each time this game is played. To make it fair, Mr. Smith would have to pay Mr. Jones this amount each time he played this game.

Exercise 13

Think of two real-life situations similar to the guessing game between Mr. Smith and Mr. Jones.

Exercise 14

What is the value of the game of odds and evens discussed in Exercise 1? Is this a fair game?

Exercise 15

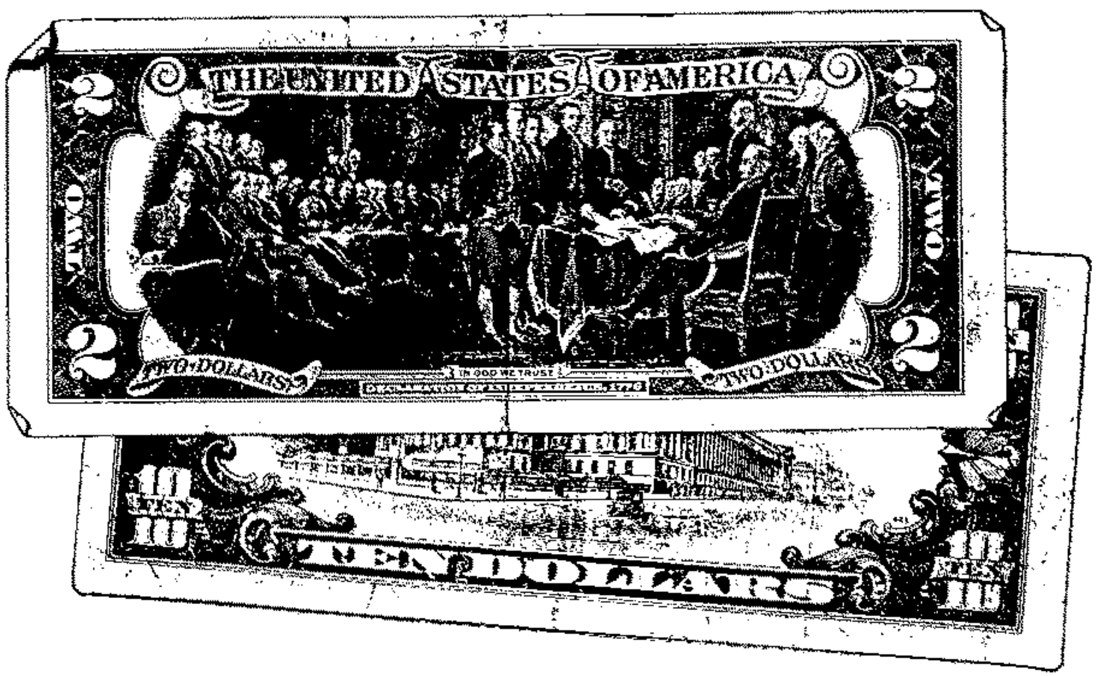
Assume that instead of a \$2 bill, Mr. Jones hides a \$10 bill. What are the optimal mixed strategies of the players in this game? How much would Mr. Smith have to pay Mr. Jones to make this game fair?

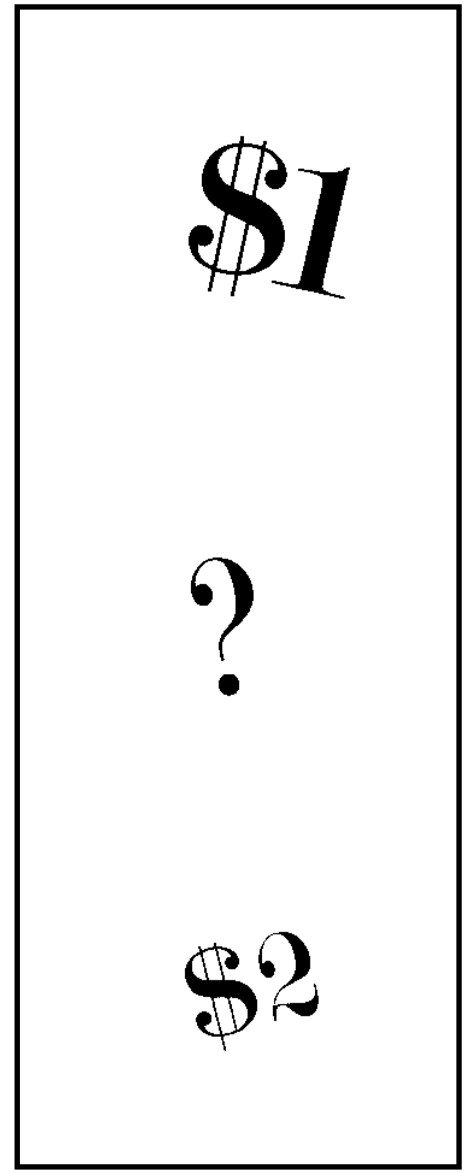
Exercise 16

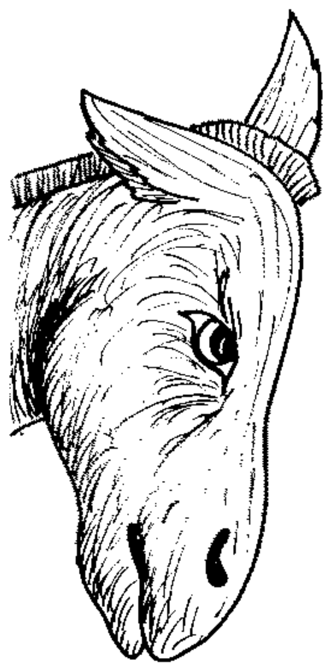
An interesting problem is to find the value of games such as

	Player B	
Player A	4	−5
	−5	6

Many people think games like this one are fair since 4 + 6 = (–5) + (–5). Find the value of this game and show that it is not fair.







Section 5. Nonzero-Sum Games

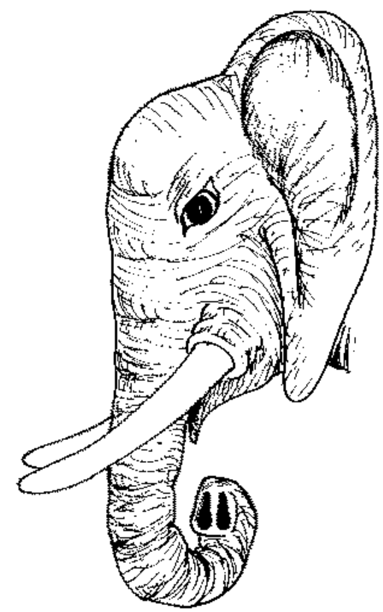
Up to this point we have been dealing with zero-sum games, or games of total conflict. But conflict does not have to be complete and, indeed, it rarely is, as the game depicted in Figure 5 illustrates. This game describes an actual conflict between Democrats and Republicans in 1981 in the House of Representatives over the implementation of President Reagan's economic program for the upcoming fiscal year.

		Republicans	
		Support Reagan Completely	Compromise
Democrats	Mainly Support Reagan	Republicans win; Democrats avoid blame. (2, 4)	Republicans win but vex Reagan; Democrats share credit. (3, 3)
	Attack Reagan	Republican program blocked in House; Democrats blamed. (1, 2)	Republicans lose much of program; Democrats look fiscally responsible. (4, 1)

Figure 5

The strategies of the players and the associated outcomes—as given by an economic columnist for the *New York Times* [7]—are described verbally in Figure 5. Two numbers are used to represent the "payoffs" to the players in this game. By convention, the first entry in each cell of the payoff matrix represents the payoff to the column player (here, the Republicans).

The payoff outcomes are ranked from 4 to 1, with "4" representing each player's best outcome, "3" each player's next-best outcome, and so on. since the payoffs in each cell are only ranks, they clearly do not sum to zero. But even if the payoffs were in dollars, say, with some positive and some negative payoffs, there might be situations where both players gain or both players lose. Games like these, where the interests of the players are not diametrically opposed are called nonzero-sum games. For instance, both players can do relatively well if the Republicans compromise and the Democrats support the President, and both do less well if the Republicans support the President's program and the Democrats attack it. In the first instance, both players receive their next-to-best payoff. In the latter, the worst outcome for the Democrats and the next-to-worst outcome of the Republicans occurs.



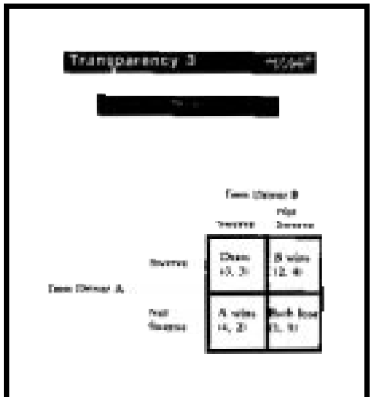
What strategies should the players select in this game if they wish to maximize their payoff? To answer this question, observe that the Republican strategy of supporting the President is dominant. This means that their payoff from their support strategy is always higher than their payoff from their compromise strategy, no matter what strategy is selected by the Democrats. For instance, if the Democrats support the President, the Republicans get their best outcome by also supporting the President, and only their next-to-best outcome by compromising. And if the Democrats attack the President's program, the Republicans get their next-to-worst outcome by supporting the President, but their worst outcome by compromising. Thus, regardless of what the Democrats do, the best strategy of the Republicans is to support the President.

Exercise 17
Determine whether either of the players in the game of Figure 1 has a dominant strategy.

By contrast, the Democrats do not have a dominant strategy in this game. If the Republicans support the President, their best strategy is to also support the President; but if the Republicans compromise, their best strategy is to compromise. Herein lies the key to analyzing this game. Given the fact that the Republican support strategy is dominant, one would expect that the Republicans would use it—and they did. And given that the Republicans supported the President, the best response

Lesson 2. Nonzero-Sum Games

The purpose of this lesson is to demonstrate that not all games are strictly competitive (zero-sum). Students might be shown Transparency 1 again and asked to discuss why it is a zero-sum game. Then they might be shown Transparency 3 which illustrates the game of Chicken. After discussing the notational differences between t he two games [i.e., only one payoff in the zero-sum game of Transparency 1, and 2 payoffs in the nonzero-sum (Chicken) game], they might be asked to identify the common interests of the players in Chicken [to avoid outcome (1, 1)]. Students might be asked to imagine other games in which the players have an incentive to cooperate with each other rather than exploit one another.



of the Democrats was to also support the President's program—and they did. The resulting outcome, (2, 4), was next-to-worst for the Democrats and best for the Republicans.

Observe that outcome (2, 4) is an equilibrium outcome. Once the players select the strategies associated with it, neither has an incentive to switch to another strategy, as long as the other player also does not switch strategies. For instance, if the Democrats switch from their support to their attack strategy, they would move from their next-worst (2) to their worst (1) strategy, they would move from their best (4) to their next-best (3) outcome. The stability of this outcome explains why once the players selected their strategies when this game was actually played they did not subsequently change them.

Exercise 18

Are any other outcomes in equilibrium in the game of Figure 5?

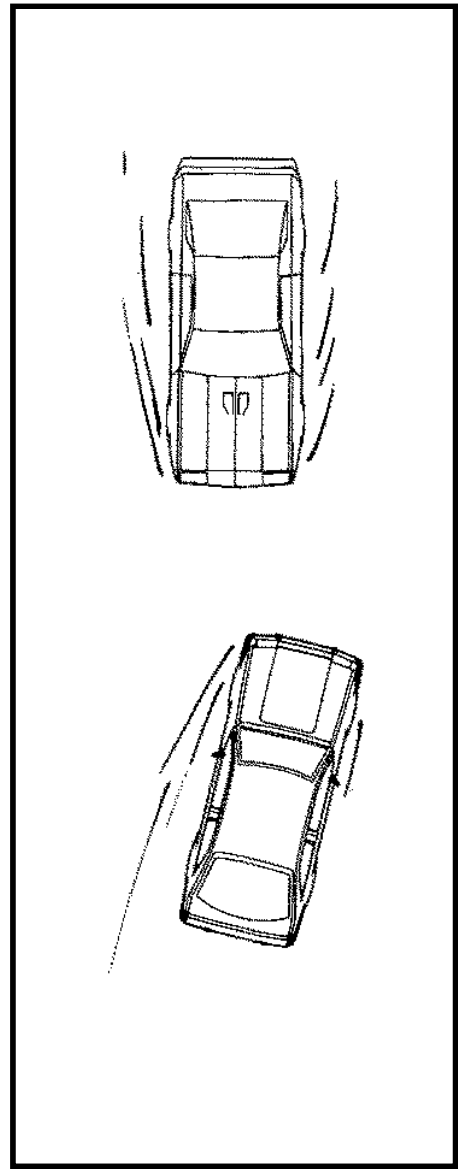
Since all zero-sum games have a well-defined solution, whether or not they have a saddlepoint, and since the concept of an equilibrium outcome provides a neat explanation of the actual behavior of the players in the economic policy game Figure 5, one might expect that the analysis of nonzero-sum games. This, however, is not the case. In some games, there may be no easily identified solution, even when equilibrium outcomes exist. In other games, rational choices may lead to counter-intuitive conclusions. While one might be disappointed by this development, it is no doubt the advantage of formal mathematical analysis that such a pathological situation can be identified and better understood.

To see this, consider now the game listed in Figure 6. In this game, known as "Chicken," each player drives an automobile at a high speed on a narrow road toward each other. Each player can either swerve or not swerve. If one swerves and the other does not, the player who swerves loses (face) while the player who does not swerve wins by demonstrating his courage. If both swerve, neither loses. And if neither swerves, both lose more than their face. These considerations are reflected in the ranking of the outcomes—from 4 to 1, with "4" being best, and so on—in the matrix of Figure 6.

		Driver B	
		Swerve	Not Swerve
Driver A	Swerve	Draw (3, 3)	B wins (2, 4)
	Not Swerve	A wins (4, 2)	Both lose (1, 1)

Figure 6

There are two equilibria in this game, (4,2) and (2,4). However, they do not help us to identify optimal strategies for the players. Neither player has a dominant strategy. Each player's best strategy depends upon the strategy selected by the other player. For instance, if B swerves, A should not swerve. And if B does not swerve, A should swerve. Thus, it is impossible to say what the players in this game should do without knowing what strategy the other player is going to select. Games such as this are of more than "academic" interest. Many political scientists, for instance, have modeled the Cuban missile crisis using the game of Chicken.



Exercise 19
Explain why it might be difficult for the players to tacitly compromise and not swerve in this game.

Exercise 20
Explain the thought process that might lead both players to decide to not swerve in this game.

Exercise 21
The structure of the game of Chicken is often compared to that of an international crisis. Discuss the soundness of this analogy with either a hypothetical or real-life example such as the Cuban missile crisis.

Lesson 3. Prisoners' Dilemma

The purpose of this lesson is to demonstrate that even when players have an incentive to cooperate with each other, cooperation might be difficult. Tell the students the story of the Prisoners' Dilemma and show them Transparency 4. Ask them what they would do if they were Prisoner A (or B). Demonstrate that both players are always better off by confessing and show that *both* are nevertheless worse off if they confess. Ask students to think about an arms race and show why many political scientists compare arms races to a Prisoners' Dilemma game. (See Transparency 5.) Ask students to think of other situations where there is a conflict between individual and collective interests. discuss how consumers might benefit if two producers are involved in a price war (for instance, double coupons at the supermarket). Ask whether there should be a law against such games as some analysts have argued.

Next, consider now the following situation: two suspects are taken into custody. The district attorney is convinced that they are guilty of a certain crime but does not have enough evidence to convince a jury. Consequently, he separates the suspects and tells each one that he has two choices: to either confess or not to confess to the crime. The suspects are told that if both will confess, neither will receive special consideration, therefore, and will receive a jail sentence of five years. If neither confesses, both will probably be convicted of some minor charge and have to spend one year in jail. But if one confesses and the other does not, the suspect who confesses will be set free for cooperating with the state while the suspect that does not will have the book thrown at him and receive a ten-year jail sentence. These considerations are reflected in the payoff matrix of Figure 7. What should each suspect do? What would you do if you were one of the suspects and wanted to minimize the number of years you had to stay in jail?

		Prisoner B	
		Not Confess	**Confess**
Prisoner A	**Not Confess**	(−1, −1)	(−10, 0)
	Confess	(0, −10)	(−5, −5)

Figure 7

Notice from Figure 7 that each suspect's confess strategy dominates his not-confess strategy. For instance, if Prisoner *B* does not confess, Prisoner *A* gets set free if he confesses, but gets one year in jail if he does not confess. And if Prisoner *B* confesses, Prisoner *A* gets five years in jail by confessing and ten years in jail by not confessing. Thus, Prisoner *A* is always better off confessing than by not confessing.

Exercise 22

Demonstrate that Prisoner B's confess strategy dominates his not confess strategy.

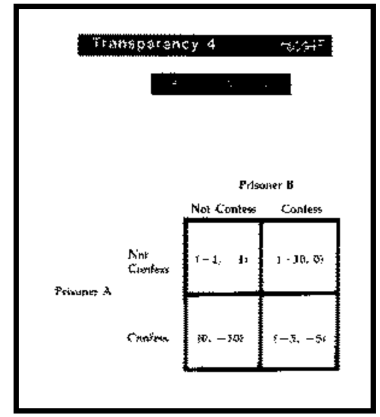
Exercise 23

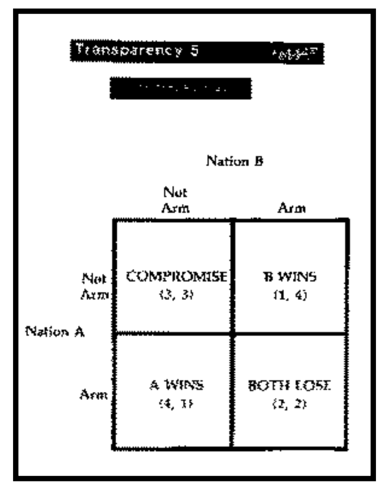
Identify the unique equilibrium outcome in the game of Figure 7. Explain why at least one equilibrium outcome will always be associated with each dominant strategy in a game.

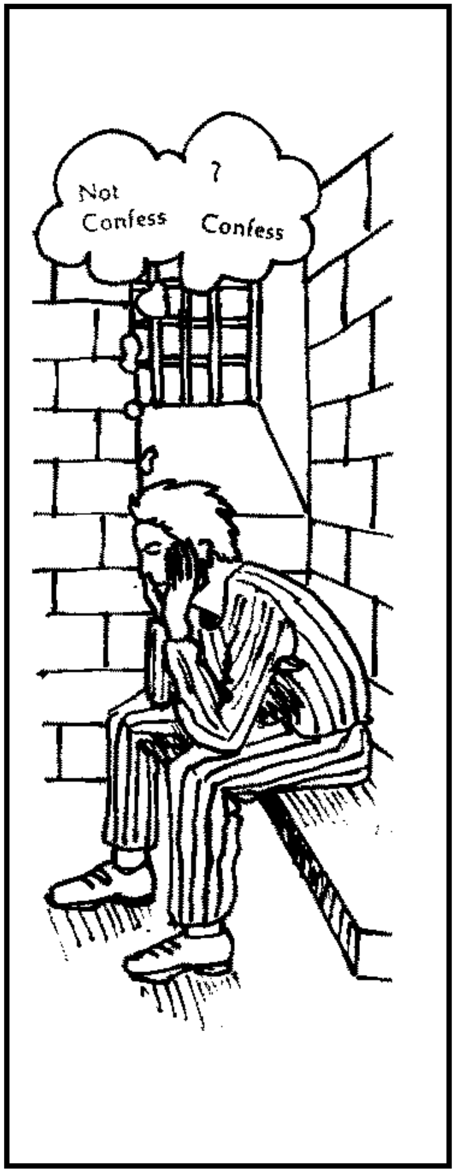
The game in Figure 7 is game theory's most famous game. Its name is "Prisoners' Dilemma." The reason why this game is characterized as a dilemma is simple. Both players have dominant strategies. Dominant strategies are unconditionally best strategies. Yet if *both* players use their optimal strategies, *both* are worse off than if they use their non-optimal strategies. If both players confess, they get five years in jail. But if both do not confess, they get only one year in jail. Paradoxically, however, since each suspect has a dominant strategy, it remains true that each suspect is *individually* better off using it and confessing.

The Prisoners' Dilemma game has spawned a great deal of theoretical and experimental research. There is good reason for this. The universal conflict between individual and collective interests, highlighted and neatly summarized by this game, lies at the heart of many important real-life situations with implications for political, social, and other kinds of systems. Many economists, for instance, hold the structure of this game responsible for phenomena such as price wars and trade barriers. And international relations specialists frequently associate the logic of arms races with that of a Dilemma game. In such situations, players may find themselves in a "catch 22" situation in which they are done in by their own rational calculations. Even though they are both better off if they cooperate, the irrefutable logic of a dominant strategy dictates that each player, in pursuing his own selfish ends, defects from cooperation. Moreover, the logic of this game shows how the invisible hand mechanism associated with *laissez-faire* economics may break down.

Psychologists have also been interested in Prisoners' Dilemma. Do adult men play this game against adult men differently from the way they would play against women or children? Questions such as this have been studied extensively in experimental situations [6].







Exercise 24

Explain why paying one's union dues or income tax may place an individual in a multi-player Prisoners' Dilemma game.

Exercise 25

Can you think of another situation that bears a resemblance to a Prisoners' Dilemma game?

The original story used to illustrate the logic of the Prisoners' Dilemma game assumes that the prisoners were separated and, hence, unable to communicate with each other. To what extent does the dilemma depend upon the inability of the players to talk with each other? Is the dilemma dissipated if the prisoners are able to communicate?

Unfortunately, for the prisoners at least, their dilemma does not evaporate even if they are allowed to negotiate a joint strategy. This is because the compromise outcome (–1, –1), which would result if neither player confessed, is unstable and not an equilibrium outcome. In other words, even if the two prisoners could agree not to confess, each would have an incentive, unilaterally, to break this agreement. Prisoner A, for instance, would go from one year in jail to no years in jail if he reneges on the agreement and turns state's evidence. Similarly, Prisoner B also has an incentive to double-cross his partner in crime. And if both confess, both end up in jail for five years. This, of course, is what the District Attorney probably had in mind all along.

Exercise 26

Identify the equilibrium strategies of the players in the following games. Which players have dominant strategies? Which games have problematic structures?

(2, 1)	(0, 0)
(0, 0)	(1, 2)

(3, 3)	(2, 4)
(4, 1)	(1, 2)

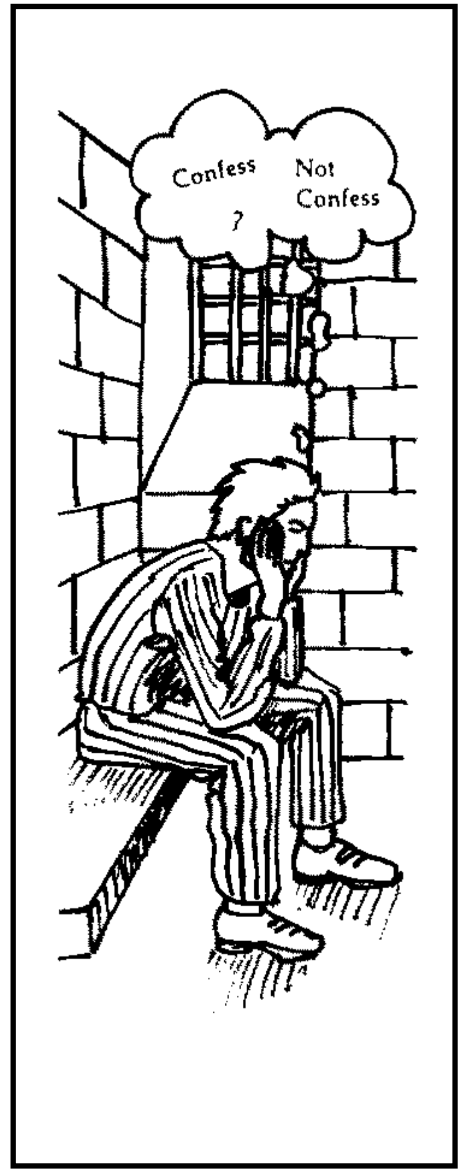
Exercise 27

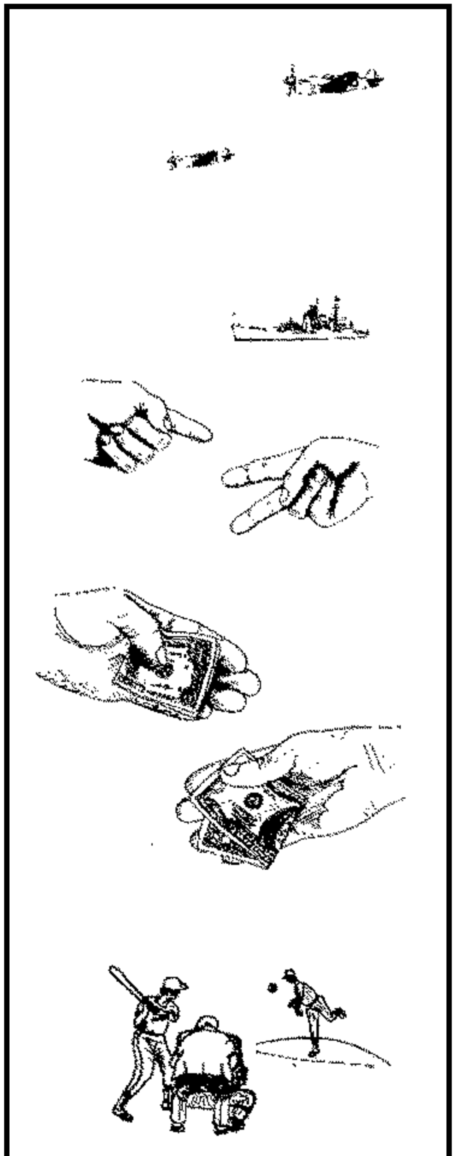
Identify the equilibrium outcomes in the following game. What strategy should each player select? Under what conditions should a player eschew an equilibrium strategy?

(4, 4)	(2, 2)
(1, 1)	(3, 3)

Exercise 28

Construct an outcome matrix for a nonzero-sum game-like decision you are familiar with and analyze it using the concepts of a dominant strategy and/or and equilibrium outcome.



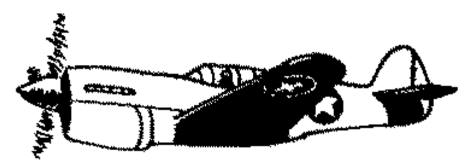


Section 6. Conclusions

This module shows that a wide range of social situations are amenable to rigorous analysis using mathematics that are well within the limits of a high school curriculum. In this module, interesting and provocative situations were analyzed. The mathematics required to participate in the discussion included:

1. Simple factoring,
2. Adding, subtracting, multiplying, dividing, and reducing fractions, and
3. Rudimentary concepts of probability.

New concepts that were introduced include the idea of an outcome matrix, a dominant strategy, an equilibrium outcome, and a mixed strategy. Game theory provides both a mechanism for introducing or reinforcing these mathematical subjects, as well as for demonstrating the benefits of rigorous analysis for understanding real-life conflicts at the personal level and beyond.



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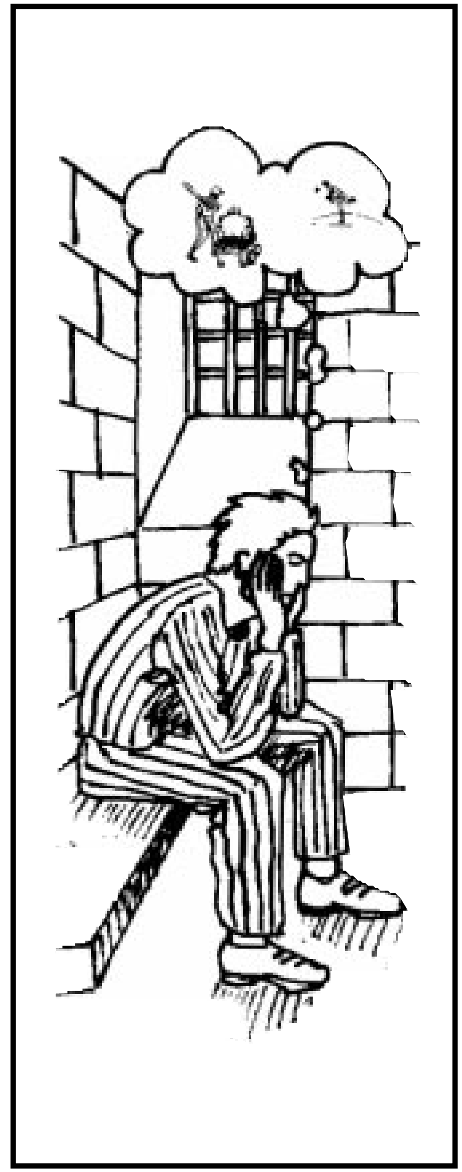
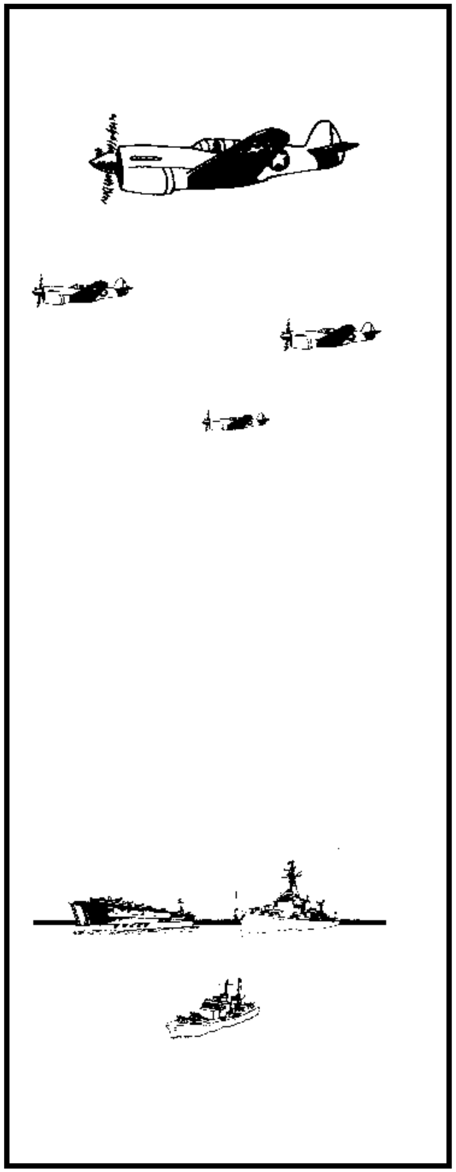




Answers to Exercises

1.

		Mr. Odd I	Mr. Odd II
Mr. Even	I	$5	–$5
	II	–$5	$5

2.

		Mr. Odd I	Mr. Odd II
Mr. Even	I	$5	–$5
	II	–$5	$5
	III	$5	–$5

Nine outcomes (3 x 3 = 9) are possible if both players can use three fingers.

3. Because the Japanese can induce a better outcome (i.e., one day of being bombed) by switching from their strategy associated with this outcome.

4. No. There is only one equilibrium in this game.

5. If the Japanese had sailed south and Kenney had searched north, their payoff would have been the same. But if Kenney had searched south, their payoff would have been even worse. One could disagree with Haywood by arguing that the Japanese should have decided not to reinforce the island.

6. Kenney's optimal strategy would have been to search south. The Japanese should continue to sail north in order to maximize their security level.

7. The assertion is true.

8. The assertion is true.

9. The pitcher does not want the batter to know the type and location of the pitch. The batter would like to know this information. The batter does best when he knows what type of pitch is coming. The pitcher does best when the batter does not know this. By mixing his pitches randomly, first a curve ball, then maybe a slider or fastball, a pitcher may be able to keep a batter from guessing what pitch he will throw next.

10. $p_1 = p_2 = q_1 = q_2 = (1/2)$. Neither player gains an advantage by knowing this probability distribution because this knowledge does not enable a player to increase his expected payoff beyond the level he can ensure by selecting his own optimal mixed strategy. But if a player knows his opponent's choice on any one play of the game, he can exploit this information and select a strategy that guarantees that he will win $5.00.

11. $p_1 = (1/3)$; $p_2 = (2/3)$; $q_1 = (1/6)$; $q_2 = (5/6)$

12. $p_1 = (2/3)$; $p_2 = (1/3)$; $q_1 = (1/3)$; $q_2 = (2/3)$

13. A lottery; a life insurance policy.

14. The value of this game is zero. It is a fair game.

15. $p_1 = (10/11)$; $p_2 = (1/11)$; $q_1 = (10/11)$; $q_2 = (1/11)$. The value of this game is 10/11 or $.91. Mr. Smith would have to pay Mr. Jones this amount to make this a fair game.

16. The value of this game is –1/10.

17. The Japanese sail south strategy is dominant.

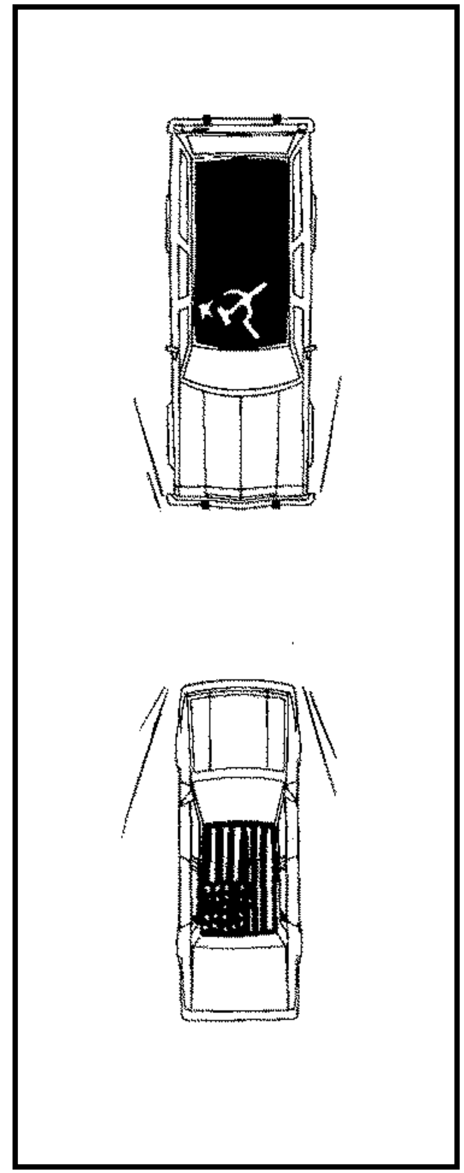
18. No.

19. (3, 3) is not an equilibrium outcome. Each player would have an incentive to upset a tacit agreement to compromise.

20. If each player thought that the other was going to swerve, his best strategy would be to not swerve.

21. Because of its structure, Chicken has often been offered as a model of an international crisis where each state is faced with the dilemma that aggressive pursuit of its national interest may result in its worst outcome, i.e., war. The Cuban missile crisis of 1962, the Berlin crisis of 1948, and the Falkland/Malvinas crisis of 1982, among others, have been depicted as Chicken games.

22. If Prisoner A does not confess, Prisoner B gets one year in jail by not confessing, but gets set free by confessing. If Prisoner A confesses, Prisoner B gets ten years in jail by not confessing, and only five years in jail by confessing.



23. (–5, –5). If one player has an unconditionally best strategy, the best response to this strategy by the other player will always be an equilibrium. The player with the dominant strategy will, by definition, not have an incentive to switch to another strategy. The other player will also lack such an incentive since the strategy associated with the equilibrium outcome is already his best response to his opponent's dominant strategy.

24. Each taxpayer (worker) gets the benefits of paying his taxes (dues) whether or not he pays his. But if no one pays, all are worse off.

25. The decision whether or not to conserve electricity during a power shortage is essentially a multi-player Prisoners' Dilemma game.

26. In the first game, (2, 1) and (1, 2) are equilibria. Both strategies are therefore equilibrium strategies. In the second game, (2, 4) is the unique equilibrium outcome. The first strategy of the row player and the second strategy of the column player are equilibrium strategies. The first game, with two equilibria, presents a problematic solution.

27. (4, 4) and (3, 3) are equilibria. A player might forego an equilibrium strategy when another equilibrium strategy provides the opportunity for a better outcome. In this game, both players might reason that the other will use his first strategy and thereby try to induce his best outcome by also selecting his first strategy.

28. A man and a woman must decide between going to a prize fight and an opera on a date. The man prefers the fight if the woman goes with him. He next prefers an opera if the woman goes with him. He would be miserable if he were at the fight or the opera alone. The woman's preferences merely reverse the preferences of the man for going to the opera or the fight when the man goes along. The first matrix listed in Exercise 26 provides and outcome matrix for this game.

Worksheet 1: Lesson 1 HiMAP

1. Determine the equilibrium outcome and the optimal strategy for each of the following games:

a)

0	2
−1	5

b)

0	0
0	−4

c)

1	4
2	3

d)

9	3
0	1

2. Find the optimal mixed strategy for each player and the value of the following games:

a)

7	−6
5	8

b)

1	0
−1	2

c)

1	−1
−1	1

d)

−1	0
0	−2

Answers to Worksheet 1: Lesson 1

1a. 0

1b. 0

1c. 2

1d. 3

2a. $p_1 = 3/16$; $p_2 = 13/16$; $q_1 = 7/8$; $q_2 = 1/8$; $v = 43/8$.

2b. $p_1 = 3/4$; $p_2 = 1/4$; $q_1 = 1/2$; $q_2 = 1/2$; $v = 1/2$.

2c. $p_1 = 1/2$; $p_2 = 1/2$; $q_1 = 1/2$; $q_2 = 1/2$; $v = 0$.

2d. $p_1 = 2/3$; $p_2 = 1/3$; $q_1 = 1/3$; $q_2 = 2/3$; $v = -(2/3)$.

1. Determine the equilibrium outcome for each of the following games:

a)

(4, 4)	(2, 3)
(1, 1)	(3, 2)

b)

(2, 4)	(4, 3)
(1, 1)	(3, 2)

c)

(3, 4)	(2, 1)
(1, 2)	(4, 3)

d)

(2, 4)	(4, 1)
(1, 1)	(3, 2)

Answers to Worksheet 2: Lesson 2

1a. (4, 4); (3,2)

1b. (2, 4)

1c. (3, 4); (4, 3)

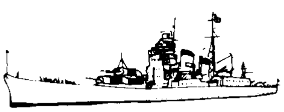
1d. (2, 4)

The Game of Odds and Evens

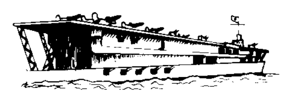
1.

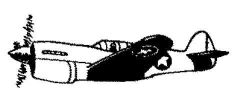
		Mr. Odd I	Mr. Odd II
Mr. Even	I	$5	–$5
Mr. Even	II	–$5	$5

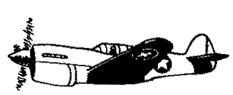
The Battle of the Bismark Sea



Japanese Commander







General Kenney	Sail North	Sail South
Search North	2 days	2 days
Search South	1 day	3 days

Chicken

		Teen Driver B	
		Swerve	**Not Swerve**
Teen Driver A	**Swerve**	**Draw (3, 3)**	**B wins (2, 4)**
	Not Swerve	**A wins (4, 2)**	**Both lose (1, 1)**

Transparency 4 HiMAP

Prisoner's Dilemma

		Prisoner B: Not Confess	Prisoner B: Confess
Prisoner A	Not Confess	(–1, –1)	(–10, 0)
Prisoner A	Confess	(0, –10)	(–5, –5)

Transparency 5 HiMAP

An Arms Race Game

		Nation B: Not Arm	Nation B: Arm
Nation A	Not Arm	COMPROMISE (3, 3)	B WINS (1, 4)
Nation A	Arm	A WINS (4, 1)	BOTH LOSE (2, 2)