Example 1. Geometric Distribution. Assume you invest in a portfolio which historically has a median yearly total return of 10% (for example the S&P 500 Index of Stocks). Half of the yearly returns were 10% or greater and half were less than 10%. See the articles in this course for various kinds of averages. For a single year chosen at random, the probability p of 10% or greater is .5, and the probability q of less than 10% is .5. We know that in general q = 1 – p.

Let N be the number of trials of yearly returns until the first return is 10% or greater. It is known that \( P(N = n) = q^{n-1} p \) for \( n \)-1 consecutive return less than 10% until a return greater or equal 10% on the \( n^{th} \) trial occurs. This gives \( P(N = n) = .5^{n-1}(.5) \). We assume that annual returns are independent. See the Side Bar notes and Exercises for independence.

Let us examine the investment in the portfolio with successive annual returns. What is the probability \( P(N = 3) \) of the first return greater than or equal to 10% occurring on the third year? We have \( P(N = 3) = (.5^2)(.5) = .125 \). We could build the Table 1:

Table 1. We will abbreviate \( P(N=n) \) with \( P(n) \).

\( P(n = 3) = .125 \) for the first return greater or equal to 10% in year 3.
\( P(n=2) = .25 \) for the first return greater or equal to 10% in year 2.
\( P(n=1) = .5 \) for the first return greater or equal to 10% in year 1.

Example 2. For this portfolio, we assume that the proportions of annual returns which lose money with return less than 0% is .31, and the proportion for which returns are greater than or equal to 0% is .69. Similar to the S&P 500 and applying the normal distribution. Here we assume \( q \), the probability of losing money is \( q = .31 \), and \( p \), the probability of not losing money, is \( p = .69 \). We could build the Table 2:

Table 2.

\( P(n = 3) = (.31^2)(.69) = .0067 \) for the probability of the first nonnegative return in year 3.
\( P(n = 2) = (.31)(.69) = .214 \) for the probability of the first nonnegative return in year 2.
\( P(n = 1) = .69 \) for the probability of the first nonnegative return in year 1.

Example 3. Compare the probability of the first year with return \( \geq 10\% \) with the probability of the first year of return \( \geq 0\% \) for years 1, 2 and 3 in Table 3.

Table 3.

<table>
<thead>
<tr>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \geq 10% )</td>
<td>.5</td>
<td>.25</td>
</tr>
<tr>
<td>( \geq 0% )</td>
<td>.69</td>
<td>.214</td>
</tr>
</tbody>
</table>
Thus the probability (.0067) of having to wait until the third year for the first $\geq 0\%$ is less than the probability (.125) of having to wait until the third year for the first $\geq 10\%$. See the Exercise for discussion of Table 3.

**Derivation** of the mean (expected value) $E(N=n)$ of the random variable $N$ for the Geometric Distribution with probability of success $p$. $N$ is the number of the trials (periods) with a first success. $P(N=n)=q^{n-1}p$ with the first success at trial $n$ where $n = 1, 2, 3, \ldots$.

It is known that $E(N) = \sum n[P(n)]$. See the Exercises to verify this.

$$E(N) = 1p + 2qp + 3q^2p + 4q^3p + \ldots$$
$$= (1-q) + 2q(1-q) + 3q^2(1-q) + 4q^3(1-q) + \ldots$$
$$= 1 - q + 2q - 2q^2 + 3q^3 - 3q^4 + 4q^5 - 4q^6 + \ldots$$
$$= 1 + q + q^2 + q^3 + \ldots$$
$$= \frac{1}{1-q} = \frac{1}{p}$$

So $E(N) = \frac{1}{p}$. See the Exercises to show that $1 + q + q^2 + q^3 + \ldots = \frac{1}{1-q}$.

**TI 83 calculation.** We will calculate $P(n=3)$ for $p = .69$ on the TI 83 function geompdf(p,n). Code and commentary: 2nd Dist $\downarrow\downarrow$ to geometpdf( Enter. You see geometpdf( Write in .69,3 to get geometpdf(.69,3. Press Enter, you interpret geometpdf(.69,3) = .066309. This should "agree" with the above value in Table 2.

Consider calculating the probability of a success occurring on years $n = 1, 2, 3$ by using the cumulative distribution function geometcdf(. Code and commentary 2nd Dist $\downarrow\downarrow$ to geometcdf( Enter You see geometcdf( Write in .69,3 Your see geometcdf(.69,3 Enter This gives geometcdf(.69,3) = .970209. You should be able to express, and calculate this sum with a scientific calculator.

**Step by Step application of the Geometric Distribution.**
1. State the problem.
2. Define a successful trial. Define a failure trial.
3. Give a probability $p$ of a success and a probability $q$ of a failure, where $p + q = 1$.
4. Define the random variable $N$ which is the number of trials until the first success. Verify or assume that the trials are random or independent.
5. Specify that $P(N = n) = q^{n-1}p$ as the probability that the first success occurs on trial $n$. This could be named Geompdf(p,n). Write and describe the numerical inputs. Input the numbers.
6. Write the numerical formula for
geometpdf(p,n) = q^{n-1} \; p = P(N=n) \text{ which gives the probability of the first success on the } n^{th} \text{ trial, or}

gemoetcdf(p,n) = P(n=1) + P(n=2) + \ldots + P(N = n) \text{ which is the probability of a first success on or before the } n^{th} \text{ trial. Solve.}

7. Write an interpretation of the solution in context, and check the conclusion.

Example 4. The Binomial Distribution: What is the probability of two success of out of 5 trials, for p, p = \frac{2}{5} = .2 and q = .8 ? We need to consider the number of combinations in which 2 out of 5 can happen. This is \( \binom{5}{2} = \frac{5!}{(2!)(3!)} = 10. \) We will use \( \frac{5}{2} \) for \( \binom{5}{2} \).

So there are 10 ways to get two successes out of 5 trials. Since there are only 10 of these, you could count them with a diagram. See the Exercises. Since these possibilities are disjoint, we can simply add the probabilities getting

\[ P(2 \text{ successes out of 5 trials}) = \frac{5}{2}(.2^2)(.8^{5-2}) = .2084 \]. See the Exercises for calculations with a scientific calculator.

In general for the binomial distribution

\[ P(x \text{ in } n) = \binom{n}{x} p^x q^{n-x} \] where p is the probability of a success and q = 1 - p is probability of a failure in a single trial, and x successes out of n trials.

Calculating \( P(x \text{ in } n \text{ on the TI83}) \), for \( P(x \text{ in } n) = \binom{n}{x} p^x q^{n-x} \) we use three parameters, n trials, p = probability of a success, and x = the number of successes.

\[ \text{binompdf}(n, p, x) = \binom{n}{x} p^x q^{n-x} \].

For calculating the probability of x or fewer successes out of n trials, we could use the cumulative distribution function \( \text{binomcdf}(n, p, x) \).

For the probability of 2 or fewer successes in 5 trials from above we use n = 5, p = .2, and x = 2. \( \text{binomcdf}(5, .2, 2) = .94208 \). You will find \text{binomcdf} by the code 2^{nd} \ Distr \ \downarrow \downarrow \ldots \). See the Exercises for calculating with a scientific calculator.

Derivation of the mean = \( E(X) \) of a binomial distribution. We can consider 1 trial with the probability of success of p and probability of failure q. We will denote a success by 1 and a failure by 0.

\[ \text{Mean} = [P(0)][0] + [P(1)][1] = q[0] + p[1] = p. \]

Consider a succession of one trial events \( X_1, X_2, \ldots X_n \). We know
E(X) = E(X₁) + E(X₂) + ... + E(Xₙ) = p + p + ... + p = np . See the Exercises for examining this claim.

Example 5. We return to the p = probability of a return less than 0% = .31 in the above Example 2. The probability of a return ≥ 0% = .69. We ask for the probability of a ≥ 0% return in 2 out of 3 trials with p = .69 and q = .31. By the binomial distribution,

P(2 successes out of 3 trials) = (3/2)p²q²⁻² = \frac{3!}{(2!)(1!)}(.69)^2(.31) = .44277  
We check:

\text{binompdf(n, p, q) = binompdf(3,.69,2) = .44277}.

Resorting to the Normal Distribution. Consider with p = .06,

P(2000 out of 35000) = \frac{(35000/2000)(.06^{2000})(.94^{33,000})}{.443} . When calculating \text{binompdf(35000, .06, 2000)} we get ERR:DOMAIN. The numbers are too large for the calculator. Assume we wanted P(n ≤ 2000 ), this would require even more calculations. We will use a Normal Distribution with mean = np = 2100 and sd = \sqrt{npq} = 44.43 . Resorting to the Normal Distribution, P(n<2000) = P[ z < \frac{2000 - 2100}{44.43} ] = P(z < -2.25) = .012. The Normal Approximation gives a good approximation if np and nq are large enough. (See Bock in References.) Bock says if np = 10 and nq = 10, or greater, you get a good approximation. See the Exercises.

Side Bar Notes:

Lee reports some Spearman Rank Correlation Tests of randomness of annual returns on a stock in which the hypothesis of randomness could not be rejected (pp.772 to 779). See References for Lee. He reports a Chi Square Test of Independence for Cross Classified Attributes (p. 513.)

Two events are said to be independent when the probability of one event is not affected by the occurrence of the other. Lee page 168, Bock p. 388. See the References. Studies have been conducted showing that in some cases, stocks that do well, continued to do well for a period of time.

People forget to help themselves. One cost of helping yourself is for financial responsibility and retirement, and knowledge of financial mathematics.

Evaluation of employers’ 401k plans: based on how quickly participants accumulate the money believed to be needed to retire comfortably. Better educated employees do better. (Kiplinger’s Personal Finance, 12/2015, p. 14)

Phishing, you see it everywhere including most openly on the internet. Victims of phishers are phools. Phishing means to defraud someone by posing as a legitimate contact. (Kiplinger’s Personal Finance, 12/2015, p. 24)

Of the 7 million people age 55 and older who are working part-time, more that 5.5 million are working for non-economic reasons. (Kiplinger’s Personal Finance, 5/2015, p. 31)
Health Savings Accounts, the double tax advantaged investment. See healthsaving.com.

This could also save on the cost of health insurance. They are called HSAs.

Taxes on REITS and Limited Partnerships, which typically pay high dividends, are complicated. But you broker may do the taxes for you. Why are the taxes complicated?

Macro effects on investments is common. For example the pullback in yield on the benchmark 10-year Treasury helped the long term government bond funds return 24.1% last year. The performance of the benchmark S&P 500 affects the performance of large cap stock funds. These are examples of macro effects. R-squared is now included in on line data. This measures the percentage of the fund’s three-year return that is influenced by the S&P 500. See the articles in this course on R-squared. (AAII Journal Feb. 2015).

Student loan debt hucksters have widespread advertising promising assistance in saving money on student loan payments. Although not government agencies, they use such terms as “federal”, “US”, “.org,” or “Student aid office.” Forty million Americans are carrying $1.4 trillion in student loan debt, and 8 million are in default. In one state 800 borrowers paid $1500 each for information which was free at studentloans.gov. Some lost much more, and were in worse shape after the service than without it. Twelve percent of college students work their way through college. (Forbes magazine, Aug. 17, 2015, p. 82).

For more financial education, see Published Research for Financial Professionals at Morningstar, Ibbotson Associates.

Seventy percent of preretirement seniors believe they are not financially ready for retirement. Forty percent expect to work until they die. Half retire earlier than they had planned. Seventy percent are counting on Social Security for most of their income.

Investments weren’t delivering what was expected, 2001-2012: for stocks 8% was expected, 2.1% was the return; for bonds, the real yield was negative. (Oppenheimer Funds)

Overcoming a 2 percent advisory fee, requires selection of a fund in the top 6%, a one in seventeen chance. Over that last 15 years, for balanced funds, the median return was 5.45 percent and the top 10 percent was 7.28%, the difference is less than 2%. (Scott Burns, Denton Record Chronicle, June 15, 2014) What would a 60/40 investment in a S&P 500 index fund and a bond index fund have done? You could look this up.

Does geopolitical instability kill stock investments? They don’t usually bring on the bear market says Jim Stack, of InvesTech Research. In eighth of 11 crises, beginning with Germany’s invasion of France in 1940, through 2003, the S&P more than recovered any losses by the six-month mark. Can you name geopolitical crises since Operation Iraqi Freedom? (Kiplinger’s Personal Finance, 7/2014, p. 30)

If you are a serious budgeter, for example, look up the Child Tax Credit, the Savers’ Tax Credit, The American Opportunity Credit, the Health Savings Account.

Real Estate Professor Baen says housing prices are increasing much faster than average income. Home ownership has dropped to 62%. About forty percent live in rentals. Of these, 80 percent live in multi-family projects. He says young people don’t have the same dreams as their parents – to accumulate a little wealth. “They don’t want more of anything.” (Denton
Record Chronicle, Dec. 12, 2015.) Discuss. He recommends smaller and less expensive houses than most new houses on the market.

**Exercises:** Show your work. Label inputs and answers. Discuss in complete sentences. When appropriate, verify independence, or just state that you assume independence.

#1. An investor makes a profit in 80% of the years in a certain investment. Assume that the successes of making a profit are independent. Find the probability that in the future in this investor (a) doesn’t make a profit the first time in the fifth year. (b) They make their first profit in the fourth year. (c) They make a profit in one of the first 3 years. (c) Calculate the answers for the above with a formula and by using a geom function on a computing device. Give your code and device. (d) Describe all the possible outcomes in the first 3 years. How many are there? Which ones meet the conditions for the Geometric distribution.

#2. For a normal distribution with mean return = 10% and a standard deviation of 20%, what is the probability of a return greater than or equal to 0%, less than 0%? This distribution could be similar to that for the S&P 500 index of stocks.

#3. Build a table similar to Table 3 for <10% and <0%. Discuss the table. Is the chance of waiting until the third year for the first loss in the <10% line greater than for the first <0% in year 3?

#4. Show that Mean = E(X) = \( \sum xP(x) \). You could code failure with 0 and success with 1 and use relative frequencies.

#5. For the above derivation of E(N) = \( \frac{1}{p} \), show that \( 1 + q + q^2 + q^3 + ... = \frac{1}{1-q} \).

When is this sum finite?

#6. Use a diagram to count \( C_2 = \binom{5}{2} \).

#7. Calculate binomcdf(5, .2, 2) with a binomcdf function and a scientific calculator. Interpret the answer.

#8. Use relative frequencies to show that for the mean = p for a single Bernoulli trial with 1 for a success and 0 for a failure, and probability of 1 is p, and probability of 0 is q.

#9. For #8, show using relative frequencies that E(X_1) followed by E(X_2) = 2p.

#10. Consider 10 successes out of 50 trials with p = .20. Calculate P( 10 success out of 50 trials). Calculate P( 10 or fewer out of 50 trials). Use the Normal Distribution for an approximation by using normalcdf. How close is the approximation? How large were np and nq?

#11. For #10, consider 12 successes out of 50 trials. Calculate P( 12 successes out of 50 trials). Calculate P( 12 or fewer successes out of 50). Use the Normal Approximation. How close is the approximation? How large were np and nq?
#12. A certain insurance salesman sees an average of 5 customers in a week. Each
time he speaks to a customer, he has a 30 percent chance of making a deal.  (a) What is the
probability that he makes 5 deals after speaking with 5 customers in a week?  (b) Calculate
binompdf for each of x = 0, 1, 2, 3, 4, 5. What should their sum be? What is the shape of the
distribution? (c) Calculate binomcdf(5, .3, 5). What should it be? (d) Calculate binomcdf(5,
.3, 0). What should it be? (e) What is the salesman’s average sales per week? Interpret in terms
of discrete number of visits.  (f) Calculate the standard deviation. What does it mean? Interpret
it as discrete. State that you assume independence.

#13. Do the math. Buffett started Berkshire Hathaway in 1964, after buying a failing
textile company. Since then, an investment of $10,000 has grown to more than $100 million.
You can look up investing in Berkshire Hathaway on finance.yahoo.  (Kiplinger's Personal
Finance, 12/2015, p. 22)

#14. Can you do the math to check this T. Rowe Price research. Consider a 25-year-
old investor making $25,000 a year. She contributes 3% of her salary to her 401k and there is a
1.5% matching from her employer. Her 401k earns 7% a year until retirement at age 67. At
retirement, she will withdraw 4% of retirement funds. Along with Social Security, this will
cover only half the needed 75% of $25,000 for retirement living.

If she starts by saving 6% with a 3% match in her 401k, and increases the rate by 2
percentage points a year until she reaches 15%, along with 25% of total retirement income which
is Social Security, this allows her to maintain her preretirement income for 30 years.  (T. Rowe
Price Investor, Summer 2015)

Could this be done seriously without considering inflation, at least during the retirement
years?  Perhaps you could just use the 4% rule.

#15. Would you want to know these famous stock superstar investors and their investment
style?  You could call them famous investors. You can find them on Wikipedia:  See John
Templeton, Warren Buffett, Martin Zwieg, Benjamin Graham, David Dodd, and others.

#16. A friend has a teacher retirement pension paying $85,000 a year with no
guaranteed COLA. Like many Texas teachers has no Social Security. In a church devotional, he
criticized people for hoarding money. How much would you have to hoard to pay this pension?
Calculate from several points of view. Would you want him to teach your children?  Discuss.
Would his behavior be typical of many Texas teachers?  If he expects to live on $85,000 a year,
with inflation where will he be in 35 years?  He is a mathematics teacher and a school
administrator, and says he knows nothing about stocks. He says that students cannot balance a
checkbook and manage a credit card. He thinks this should be the level of their education. Our
course is for students who already know common sense. In addition, how much funds would pay
for him a Social Security retirement? See the Quick Calculator at ssa.gov.

Teacher Retirement System of Texas (TRS):  For the 12-month period ending Aug. 31,
2015, the TRS portfolio had a return of -.03 percent. For the past 10-years, the time weighted
compound annual return has been 6.2 percent. On a three-year annualized basis, the fund
returned 8.3 percent. Annualized rates of return for the five and 10-year periods ending Aug. 31,
2015 were 9.6 percent and 6.2 percent respectively.
As of Aug. 31, 2015, the system had a funded ratio of 80.2 percent with an Unfunded Actuarial Accrued Liability of $33 billion. The system is now deferring net investment losses of $4.9 billion. The time necessary to amortize the unfunded liability has also increased from 29.8 years to 33.3 years. The actuary estimates that the unfunded liability and the funding period will both continue to increase over the next few years before beginning to once again decline. Discuss the technical terms (trs.texas.gov). (See in this course, “Time Weighted Return versus Dollar Weighted Return” and other articles.) (See Google.)

References:

Cheng F. Lee, Rutgers University; John C. Lee, The Chase Manhattan Bank; Alice C. Lee, MicroStrategy, Inc. Over 1150 pages, hundreds of topics. Used textbooks can be bought at amazon or barns and nobles for less that the cost of shipping. ) For fines for the mortgage crisis, see on Google, Everything you need to know about J P Morgan’s $13 billion settlements, (including The Chase Manhattan Bank). There are Financial Mathematicians who published that they saved their banks from buying these toxic mortgages and described the mathematics that was misused.


For a free course in financial mathematics, with emphasis on personal finance, for upper high school and undergraduate college, see COMAP.com. Register and they will e-mail you a password. Simply click on an article in the annotated bibliography, download it, and teach it. Unit 1: The Basics of Mathematics of Finance, Unit 2: Managing Your Money, Unit 3: Long-Term Financial Planning, Unit 4: Investing in Bonds and Stocks, Unit 5: Investing in Real Estate, Unit 6: Solving Financial Formulas for Interest Rate, For about thirteen more advance or technical articles, see the UMAP Journal at COMAP. The last section is Additional Articles on Financial Mathematics or Related to Personal Finance. In all, there are about eighty articles.