VCTAL

The Value of Computational Thinking Across Grade Levels

Computational Thinking Module

Fair and Stable Matchings

STUDENT EDITION

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Computational Thinking Module

Fair and Stable Matchings

STUDENT EDITION

By

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Funded by the National Science Foundation,
Award No. DRL-102-0201

This material was prepared with the support of the National Science Foundation.
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Published by COMAP, Inc. in conjunction with DIMACS, Rutgers University.

COMAP, Inc.
175 Middlesex Turnpike, Suite 3B
Bedford, MA 01730
www.comap.com

ISBN:
0-9971490-2-7
978-0-9971490-2-9

Fair and Stable Matchings

Glossary

algorithm An explicit repeatable process or set of rules for solving a problem.

computational thinking A way of looking at and thinking about the world in terms of how information and data can be defined, generated, collected, related, analyzed, represented, and shared using computer and other information technologies.

default The way something is done in the absence of other specifications being provided.

instance A particular example of a type or category of problems.

matching Typically, a set of paired assignments between two distinct groups of individuals or things; however, in some contexts the matches are made within one group and sometimes more than two items might be involved in the assignments making up the matching.

parameter A value that can be adjusted to suit a particular purpose in a program or process. In most web browsers, the user can decide after how many days the web history of the browser will be cleared. This value, can be changed depending on the needs of the user; the value is a parameter associated with how the web browser functions.

partial order A transitive ordering of a set that does not require all pairs of elements to be ordered; there can be "ties" in which some pairs are not orderable according to the rule used to order the set. Partially ordered set is abbreviated *poset*.

transitive A relationship is transitive if when a relation holds between *A* and *B* and between *B* and *C*, then the relation holds between *A* and *C*; for example, "less than" is a transitive relationship: if *A* is less than *B*, and if *B* is less than *C*, then *A* is less than *C*.

Name:____

Handout:

Figures for Make Me a Match Discussion

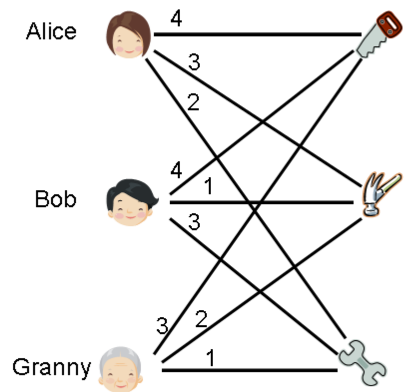


Figure 1

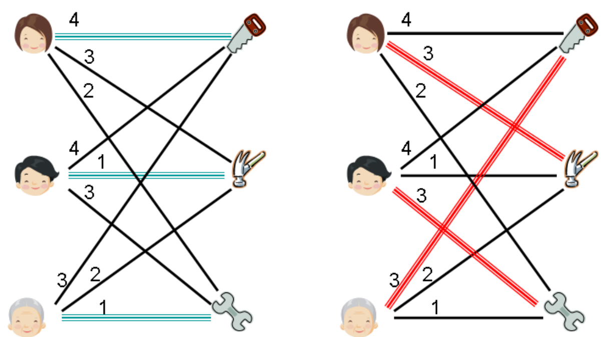


Figure 2

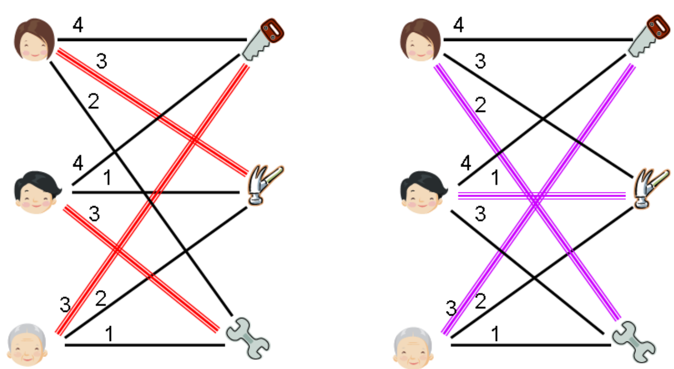


Figure 3

Name:________________________________

Handout:

Representing Preferences: Lists and Matrices—Page 1 of 2

The Big Sibling organization has collected the preferences both of the students wanting to volunteer as Big Siblings and of the Little Siblings, those younger students wanting to be matched with a Big Sibling.

Below are the two sets of lists in which everyone's first choice is listed first. For example, in reading these lists we see that both Big Siblings C and E have as their first choice Little Sibling x.

Big Sib Preference Lists

	1st	2nd	3rd	4th	5th
A:	w	y	x	z	v
B:	w	x	z	v	y
C:	x	y	w	v	z
D:	w	x	z	v	y
E:	x	y	w	v	z

Lil Sib Preference Lists

	1st	2nd	3rd	4th	5th
v:	E	A	D	C	B
w:	C	E	A	D	B
x:	A	C	E	B	D
y:	D	A	E	B	C
z:	E	C	B	D	A

What else do you notice about this way of representing the preferences?

Name:______________________________

Handout:
Representing Preferences: Lists and Matrices—Page 2 of 2

Matrices (or tables of numbers) are another way of representing all the information that is in the preference lists on page 1. In this case, the rows and columns correspond to the people we are trying to match. Each entry in the body of the matrix will indicate how one person has ranked another person. As with the preference lists, we will need two preference matrices: one for the Big Sibs' rankings of the Lil Sibs, and one for the Lil Sibs' rankings of the Big Sibs. Fill in the matrices using the preferences given on page 1.

Big Sib Preference Matrix

	v	w	x	y	z
A					
B					
C					
D					
E					

Lil Sib Preference Matrix

	A	B	C	D	E
v					
w					
x					
y					
z					

What do you notice about this way of representing the preferences?

Homework:
Finding a Good Matching

Name:______________________________

Introducing the Big Sibling/Little Sibling Case Study

You have been hired as a computational consultant by the local Big Sibs organization to help figure out the best matches between the Little Siblings and the Big Siblings. The organization collected the preferences of each Little Sibling and each Big Sibling and organized the data into preference matrices. Uppercase letters refer to the Big Siblings and lowercase letters refer to the Little Siblings.

Big Sibling Preferences

	v	w	x	y	z
A	5	1	3	2	4
B	4	1	2	5	3
C	4	3	1	2	5
D	4	1	2	5	3
E	4	3	1	2	5

Little Sibling Preferences

	A	B	C	D	E
v	2	5	4	3	1
w	3	5	1	4	2
x	1	4	2	5	3
y	2	4	5	1	3
z	5	3	2	4	1

The director of the organization has given you two tasks: (1) to evaluate a few possible matchings that the members of the organization think might be good and (2) to see if you can find any matchings that would be better than theirs.

1. Evaluate these potential matchings and explain their strengths and weaknesses.

I. {{A, x}, {B, z}, {C, v}, {D, y}, {E, w}}

II. {{E, v}, {A, w}, {B, x}, {D, y}, {C, z}}

III. {{C, x}, {A, w}, {E, y}, {B, z}, {D, v}}

2. Find another potential matching that you think the organization should consider and explain its advantages and disadvantages.

Name:____________________

Class Activity:
Testing for Stability—Page 1 of 3

Defining *Stable Matching*
A matching is *stable* if no two people *each* prefer any other to their current partners. A matching is *unstable* if at least one pair of people have *each* ranked another person higher than their current partners. Such an unstable pair is called an *instability*.

To prove that a matching is stable requires checking every person who did not get his or her first preference. Do any of the people whom a person ranked higher than his or her assigned partner also prefer that person to his or her assigned partner?

Sometimes you can quickly find an instability by focusing on the people who are assigned low preferences. Consider Big Sibling D and Little Sibling v in Matching I from the homework. D is assigned to y but prefers v; *and* v prefers D over C. Because of their *mutual* preference for each other over their current partners, Matching I is unstable. For D, 4 is better than 5, and for v, 3 is better than 4. Other instabilities might exist; however, the presence of just one instability makes a matching unstable.

Matching I

3	A	x	1
3	B	z	3
4	C	***v***	***4***
5	***D***	y	1
3	E	w	2

Mutually Preferred Match

4	***D***	***v***	***3***

Big Sibling Preferences

	v	w	x	y	z
A	5	1	3	2	4
B	4	1	2	5	3
C	4	3	1	2	5
D	4	1	2	5	3
E	4	3	1	2	5

Little Sibling Preferences

	A	B	C	D	E
v	2	5	4	3	1
w	3	5	1	4	2
x	1	4	2	5	3
y	2	4	5	1	3
z	5	3	2	4	1

Name:______________________________

Class Activity:
Testing for Stability—Page 2 of 2

With a partner, determine if Matching II from the homework is stable or unstable.

Big Sibling Preferences

	v	w	x	y	z
A	5	1	3	2	4
B	4	1	2	5	3
C	4	3	1	2	5
D	4	1	2	5	3
E	4	3	1	2	5

Little Sibling Preferences

	A	B	C	D	E
v	2	5	4	3	1
w	3	5	1	4	2
x	1	4	2	5	3
y	2	4	5	1	3
z	5	3	2	4	1

Matching II

4	E	v	1
1	A	w	3
2	B	x	4
5	D	y	1
5	C	z	2

Name:____________________________________

Class Activity:
Testing for Stability—Page 3 of 3

With a partner, determine if Matching III from the homework is stable or unstable.

Big Sibling Preferences

	v	w	x	y	z
A	5	1	3	2	4
B	4	1	2	5	3
C	4	3	1	2	5
D	4	1	2	5	3
E	4	3	1	2	5

Little Sibling Preferences

	A	B	C	D	E
v	2	5	4	3	1
w	3	5	1	4	2
x	1	4	2	5	3
y	2	4	5	1	3
z	5	3	2	4	1

Matching III

1	C	x	2
1	A	w	3
2	E	y	3
3	B	z	3
4	D	v	3

Name:______________________________

Homework:
A Method for Finding Stable Matchings

1. Design an algorithm (a step-by-step procedure or list of instructions) for finding a stable matching. Your algorithm must be stated completely, concisely, clearly, and precisely enough for someone else to use successfully. Your algorithm must work with any two pairs of preference matrices that someone might have.

2. After you have written your algorithm, apply it to the set of preference matrices in the Big Sibling/Little Sibling case study to demonstrate how it works and to find what you think is a stable matching for this set of preference matrices.

	v	w	x	y	z
A	5	1	3	2	4
B	4	1	2	5	3
C	4	3	1	2	5
D	4	1	2	5	3
E	4	3	1	2	5

	A	B	C	D	E
v	2	5	4	3	1
w	3	5	1	4	2
x	1	4	2	5	3
y	2	4	5	1	3
z	5	3	2	4	1

Name:__________________________________

Handout:
The Gale-Shapley Algorithm—Page 1 of 2

1. One at a time and in any order, each Big Sib proposes to his or her highest ranked Lil Sib who has not rejected the Big Sib.
2. After each proposal, the Lil Sib must either tentatively accept the proposal (if he or she is unengaged) or must choose the higher ranked Big Sib (if he or she is already engaged).
3. Any Big Sib who is rejected by a Lil Sib proposes to the next Lil Sib on his or her list before another Big Sib makes a proposal.
4. Repeat steps 2 and 3 until there are no Lil Sibs entertaining competing proposals.
5. Repeat steps 1 through 4 until all the Big Sibs and Lil Sibs are engaged.

	v	w	x	y	z
A	5	1	3	2	4
B	4	1	2	5	3
C	4	3	1	2	5
D	4	1	2	5	3
E	4	3	1	2	5

	A	B	C	D	E
v	2	5	4	3	1
w	3	5	1	4	2
x	1	4	2	5	3
y	2	4	5	1	3
z	5	3	2	4	1

Big Sibling Proposals

A

B

C

D

E

Name:____________________________________

Handout:
The Gale-Shapley Algorithm—Page 2 of 2

1. One at a time and in any order, each Lil Sib proposes to his or her highest ranked Big Sib who has not rejected the Lil Sib.
2. After each proposal, the Big Sib must either tentatively accept the proposal (if he or she is unengaged) or must choose the higher ranked Lil Sib (if he or she is already engaged).
3. Any Lil Sib who is rejected by a Big Sib proposes to the next Big Sib on his or her list before another Lil Sib makes a proposal.
4. Repeat steps 2 and 3 until there are no Big Sibs entertaining competing proposals.
5. Repeat steps 1 through 4 until all the Lil Sibs and Big Sibs are engaged.

	v	w	x	y	z
A	5	1	3	2	4
B	4	1	2	5	3
C	4	3	1	2	5
D	4	1	2	5	3
E	4	3	1	2	5

	A	B	C	D	E
v	2	5	4	3	1
w	3	5	1	4	2
x	1	4	2	5	3
y	2	4	5	1	3
z	5	3	2	4	1

Little Sibling Proposals

v

w

x

y

z

Name:________________________

Homework:
The Gale-Shapley Algorithm—Page 1 of 2

1. Use the Gale-Shapley algorithm to find stable matching(s) for the preference matrices below. Label your proposals in the order they occur, as we did in class.

	s	t	u	v	w	x	y	z
A	5	8	7	1	4	6	2	3
B	3	4	6	2	5	7	8	1
C	4	1	3	5	8	2	6	7
D	7	6	8	3	2	4	1	5
E	3	5	8	6	7	2	1	4
F	8	4	1	6	5	3	7	2
G	3	6	8	2	4	5	7	1
H	2	1	7	3	4	5	6	8

	A	B	C	D	E	F	G	H
s	8	5	6	7	2	3	1	4
t	7	1	6	5	4	8	3	2
u	3	6	4	2	7	8	1	5
v	6	7	3	8	1	4	5	2
w	1	8	6	7	3	5	4	2
x	6	3	5	2	1	4	7	8
y	3	4	1	5	2	7	8	6
z	2	6	8	3	7	5	4	1

2. How many stable matchings do you think exist for the matrices above? Why?

Name:______________________________

Homework:
The Gale-Shapley Algorithm—Page 2 of 2

3. Create a pair of preference matrices with some interesting property or distinctive feature and then use the Gale-Shapley algorithm to find a stable matching (or two, if a second one exists). Explain what is special about your matrices and explain what the Gale-Shapley algorithm revealed about them.

	u	v	w	x	y	z
A						
B						
C						
D						
E						
F						

	A	B	C	D	E	F
u						
v						
w						
x						
y						
z						

4. Explain what the Gale-Shapley algorithm can tell us about a stable-matching problem.

Name:______________________________

Case Study:
Which Stable Matching Is Fairest of All?—Page 1 of 4

Hospitals and medical students are keenly interested in stable matchings. The National Resident Matching Program (NRMP) matches hospitals with medical students for their residencies. In 1998, the NRMP modified its matching algorithm because of an issue of fairness. Typically, many stable matchings are possible and determining a "best" one means deciding whether the hospitals' preferences or the medical students' preferences should be given priority when comparing the options. In this case study, we are going to assume that "best" means fairest. We want to treat the hospitals and the students equally. Deciding what is fairest involves working with large and complex mathematical structures. In 1962, David Gale and Lloyd Shapley initiated the formal study of stable matchings and some of their early questions remain unanswered. How will you define fairness?

You work for a Computational Consulting Company (come up with a better name than C3) that has been asked to determine which stable matching is fairest for a particularly complex problem facing a small country with eight hospitals and eight medical students. One student must be placed at each hospital and the matching must be a stable one.

Your task is to create a *measure of fairness* that can be used to compare many stable matchings and select the one that you believe is fairest. You will need to define your way of measuring fairness clearly enough so that another person could apply your method and get the same results. Different consulting companies will likely come up with different measures, depending on the consultants' beliefs about what is fair.

The presidential cabinet of the country (led by your teacher) will decide which company's approach is best and will award a lucrative contract to the winning proposal. Be prepared to explain and defend your method of measuring fairness.

The data follow on the next two pages.

Name:______________________

Case Study:
Fairest of All—Preference Matrices—Page 2 of 4

Below are the preference matrices for the hospitals (A through H) on the left, and the residents (s through z) on the right.

Hospitals' Rankings of Residents

	s	t	u	v	w	x	y	z
A	5	8	7	1	4	6	2	3
B	3	4	6	2	5	7	8	1
C	4	1	3	5	8	2	6	7
D	7	6	8	3	2	4	1	5
E	3	5	8	6	7	2	1	4
F	8	4	1	6	5	3	7	2
G	3	6	8	2	4	5	7	1
H	2	1	7	3	4	5	6	8

Residents' Rankings of Hospitals

	A	B	C	D	E	F	G	H
s	8	5	6	7	2	3	1	4
t	7	1	6	5	4	8	3	2
u	3	6	4	2	7	8	1	5
v	6	7	3	8	1	4	5	2
w	1	8	6	7	3	5	4	2
x	6	3	5	2	1	4	7	8
y	3	4	1	5	2	7	8	6
z	2	6	8	3	7	5	4	1

The Gale-Shapley algorithm produces these two stable matchings:

Hospitals Propose

```
1.5    4.875

1 A    v 6
3 B    s 5
2 C    x 5
2 D    w 7
1 E    y 2
1 F    u 8
1 G    z 4
1 H    t 2
```

Residents Propose

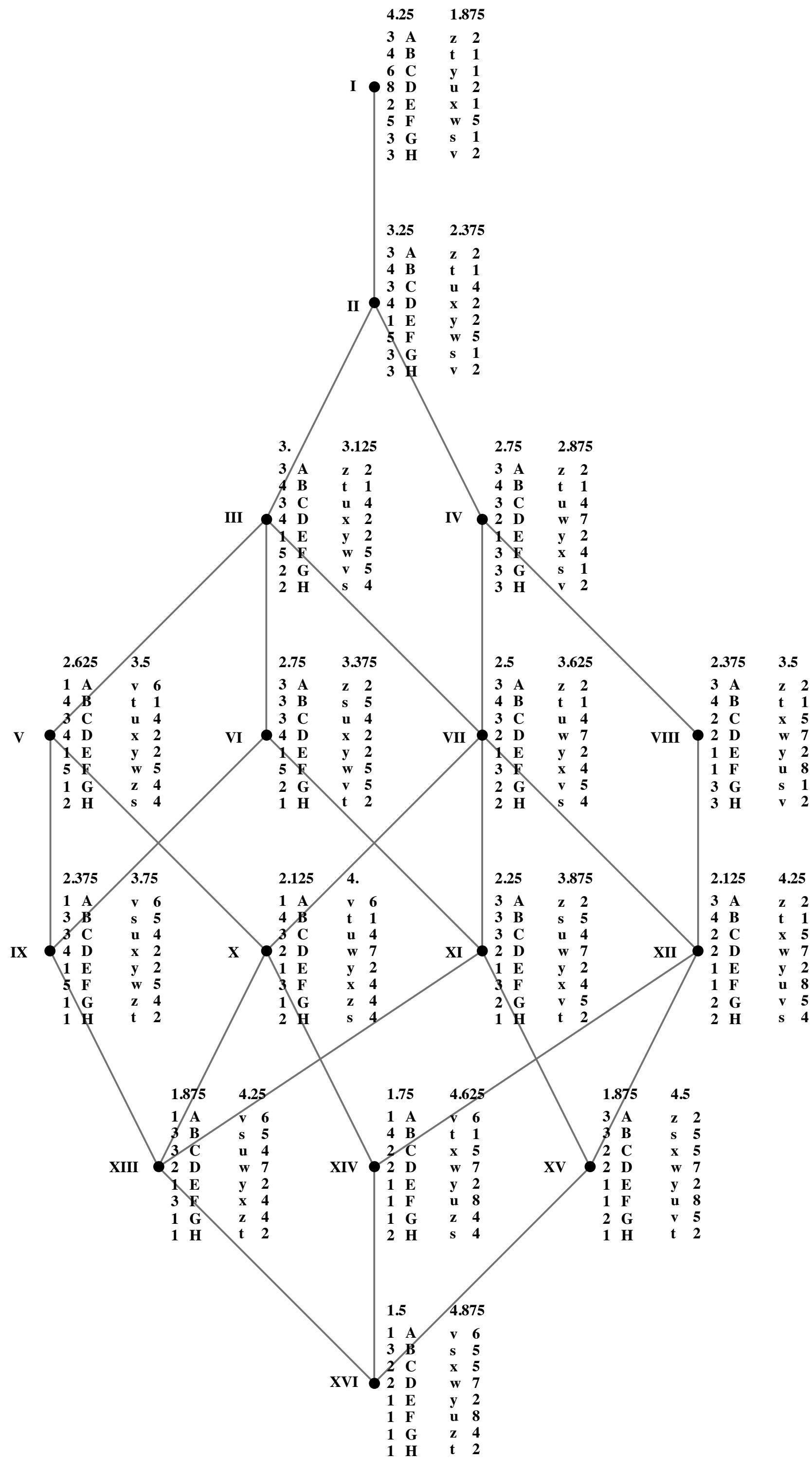
```
4.25   1.875

3 A    z 2
4 B    t 1
6 C    y 1
8 D    u 2
2 E    x 1
5 F    w 5
3 G    s 1
3 H    v 2
```

The best possible result for the hospitals has an average preference of 1.5 and the best possible result for the residents has an average of 1.875. But there are *fourteen* more stable matchings to consider! All sixteen stable matchings for the two preference matrices above are on the following page. Your task is to devise a way to determine which one is fairest overall. How would you measure fairness?

The stable matchings on the next page have been *partially ordered*. A line connects two matchings if every hospital (or resident) has at least as good a partner in one matching as it does in the other matching.

Name:______________________________

Case Study: Fairest of All—Stable Matchings—Page 3 of 4



Name:____

Case Study:
Fairest of All—Your Report—Page 4 of 4

On separate sheets of paper, your consulting company must respond to the following prompts.

1. Explain how to calculate your measure of fairness. Provide instructions that anyone could use on any set of stable matchings, and be sure to specify how any ties should be broken. Be complete, concise, clear, and precise.

2. Provide a worked example of how to use your measure of fairness to calculate the fairness value of one matching.

3. Provide the fairness values of all the matchings, I through XVI, and sort them from fairest to least fair.

4. Which stable matching do you think is fairest? Briefly explain why.

5. Be prepared to explain why your choice is fairest and why it is better than some of the other options.

Case Study A:
Hospitals Accepting Multiple Residents (2 Problems)

As medical students complete medical school they enter a residency during which they practice medicine in a hospital under the guidance of doctors. Typically, each medical student must complete one residency at one hospital; however, some hospitals can accept several residents. For example, a large city hospital might have a dozen or more different residents, while a small rural hospital might only accept one resident. This type of matching is a *many-to-one* matching because more than one item can be matched with another item.

Similar to previous stable-matching contexts, the members of both groups (the residents and the hospitals) each rank all the members of the other group. So there are two preference matrices in which the 1's indicate the top choices, 2's the second choices, and so on. Now the number of hospitals is not equal to the number of residents and each hospital has a *capacity* that it would like to fill. The capacity is the number of residents it wants. How can we find an optimal matching now?

Problem 1: 9 Residents and 5 Hospitals

Hospital Capacities: **v**: 1 **w**: 2 **x**: 2 **y**: 2 **z**: 3

Residents' Rankings of Hospitals

	v	w	x	y	z
A	3	1	2	5	4
B	4	1	2	5	3
C	3	2	1	4	5
D	1	2	3	5	4
E	1	4	3	5	2
F	2	3	1	4	5
G	4	3	1	5	2
H	1	3	4	5	2
I	1	2	3	4	5

Hospitals' Rankings of Residents

	A	B	C	D	E	F	G	H	I
v	2	7	8	5	6	1	9	3	4
w	6	7	9	4	5	2	8	1	3
x	6	8	9	4	3	1	7	2	5
y	4	3	9	5	6	8	7	1	2
z	8	2	9	1	4	3	6	5	7

Case Study A:
Problem 2: 12 Residents and 4 Hospitals

Hospital Capacities: **w**: 3 **x**: 4 **y**: 3 **z**: 2

Residents' Rankings of Hospitals

	w	**x**	**y**	**z**
A	1	2	3	4
B	2	1	3	4
C	4	2	3	1
D	1	3	2	4
E	2	1	3	4
F	2	1	3	4
G	1	2	4	3
H	4	2	3	1
I	1	3	2	4
J	4	2	3	1
K	1	2	3	4
L	2	1	4	3

Hospitals' Rankings of Residents

	A	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**	**K**	**L**
w	12	9	1	11	8	7	10	2	5	6	4	3
x	11	10	3	12	5	7	8	1	6	9	4	2
y	11	10	1	12	6	5	8	2	9	7	4	3
z	12	10	3	11	6	5	9	1	7	8	4	2

Case Study B:
Hospitals Accepting Multiple Residents and Spouses–Page 1 of 2

As medical students complete medical school they enter a residency during which they practice medicine in a hospital under the guidance of doctors. Typically, each medical student must complete one residency at one hospital; however, some hospitals can accept several residents. For example, a large city hospital might have a dozen or more different residents, while a small rural hospital might only accept one resident. This type of matching is a *many-to-one* matching because more than one item can be matched with another item.

An additional complication is that many married couples want to be placed at the same hospital. Sometimes each spouse in a couple ranks the hospitals identically; however, sometimes each spouse ranks the hospitals independently, agreeing to go wherever they are assigned even if the placement is slightly better for one spouse than the other. This complication might require you to redefine stability.

Similar to previous stable-matching contexts, the members of both groups (the residents and the hospitals) each rank all the members of the other group. So there are two preference matrices in which the 1's indicate the top choices, 2's the second choices, and so on. Now the number of hospitals is not equal to the number of residents and each hospital has a *capacity* that it would like to fill. The capacity is the number of residents it wants. Furthermore, some of the residents must be assigned to the same hospital. How can we find an optimal matching now?

Case Study B:
Hospitals Accepting Multiple Residents and Spouses–Page 2 of 2

12 Residents, 3 Married Residents, and 4 Hospitals

Hospital Capacities: **w**: 3 **x**: 4 **y**: 3 **z**: 2

The symbols ***, #,** and **^** indicate married residents (B & E, D & F, and J & L). Each member of a couple must be placed in the same hospital.

Residents' Rankings of Hospitals

	w	**x**	**y**	**z**
A	1	2	3	4
B*	2	1	3	4
C	4	2	3	1
D#	1	3	2	4
E*	2	1	3	4
F#	2	1	3	4
G	1	2	4	3
H	4	2	3	1
I	1	3	2	4
J^	4	2	3	1
K	1	2	3	4
L^	2	1	4	3

Hospitals' Rankings of Residents

	A	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**	**K**	**L**
w	12	9	1	11	8	7	10	2	5	6	4	3
x	11	10	3	12	5	7	8	1	6	9	4	2
y	11	10	1	12	6	5	8	2	9	7	4	3
z	12	10	3	11	6	5	9	1	7	8	4	2

Case Study C: Roommate Matching Problem

Not all matching problems involve matching two separate groups of items. Consider a summer sports camp that houses the athletes in double rooms. Prior to arriving at the camp, the athletes view profiles of all the other athletes and provide their preferences. How should the camp directors assign roommates for an optimal matching?

Note that, unlike previous case studies in this module, in this context there is just one preference matrix in which everyone ranks everyone else.

For example, reading across the third row we see that C's first choice is D, C's second choice is L, C's third choice is A, and so on. And reading down the fourth column, we see that athlete D has been ranked first by A, B, C, and K.

	A	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**	**K**	**L**
A		2	4	1	3	6	5	8	7	9	10	11
B	4		2	1	3	7	10	9	5	8	11	6
C	3	5		1	4	6	9	7	8	10	11	2
D	1	2	3		6	8	10	9	4	7	11	5
E	3	4	11	2		1	8	10	5	7	9	6
F	7	4	3	5	8		1	2	6	10	11	9
G	7	11	4	2	1	8		3	9	5	10	6
H	9	11	10	8	1	6	7		3	2	5	4
I	5	3	4	2	1	9	6	10		7	8	11
J	9	5	6	10	2	11	8	4	7		3	1
K	9	8	4	1	2	3	10	11	5	6		7
L	4	5	6	7	8	11	10	1	9	3	2	

Final Case Study Report

Consulting Company: ____

Names: ____

Case Study: ____

Due Date: ________________

Format for Written Report and Oral Summary

1. Explain your definition of optimality and fairness for this type of problem.

2. Explain your method for finding an optimally fair matching.

3. Give the matching you recommend be used and explain its pros and cons.

In addition to providing written responses to the above prompts, prepare to give a 5-minute oral summary. The summary should convince the organization that your approach and solution are the best.